LOWER DIMENSIONAL APPROXIMATION OF THIN ELASTIC AND PIEZOELECTRIC SHELLS

A thesis submitted
in partial fulfillment for the degree of

Doctor of Philosophy

by

J. RAJA

Department of Mathematics
INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
Thiruvananthapuram - 695 547

July 2014

CERTIFICATE

This is to certify that the thesis titled **Lower Dimensional Approximation of Thin Elastic and Piezoelectric Shells**, submitted by **J. Raja**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Dr. N. Sabu
Supervisor
Associate Professor
Department of Mathematics

Place: Thiruvananthapuram
July 2014

DECLARATION

I declare that this thesis titled **Lower Dimensional Approximation of Thin Elastic and Piezoelectric Shells** submitted in fulfillment of the Degree of Doctor of Philosophy is a record of original work carried out by me under the supervision of **Dr. N. Sabu**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgments have been made wherever the findings of others have been cited.

J. Raja

SC10D012

Place: Thiruvananthapuram

July 2014

ACKNOWLEDGEMENTS

It is my privilege to express my sincere gratitude to my advisor Dr. N. Sabu for his continuous support of my Ph.D studies and research; for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me throughout my research and also in the writing of this thesis.

Besides my advisor, I would like to thank the rest of my Doctoral committee members: Prof. Raju K. George, Dr. Neela Nataraj, Dr. M. P. Rajan, Dr. E. Natarajan and Dr. P. Raveendranath for their encouragement and insightful comments.

I would like to thank Director, IIST, Dr. K. S. Dasgupta, the former Director, IIST, Dr. B. N. Suresh and the Dean R & D, (IIST) for providing me this opportunity. I thank each and every member of IIST for all their help and good will. I also take this opportunity to thank IIST and ISRO for the financial support during my research work. Here, I like to express my appreciation to all the library and administrative staff at IIST for their timely help.

In retrospection, this is also an occasion for me to thank each and every one who has taught me mathematics at various levels for their motivating lectures. Especially I would like to thank Prof. Shanmugam, Prof. S. Gowrinathan, Prof. M. Marudai, Dr. Sushama N. Agarwal, and Dr. V. Indumathi for encouraging me and initiating me into research in mathematics.

I would like to thank my uncle Mr. M. Sri Rengan who inspired me to take up Mathematics and his sons Mr. S. Muthu Krishnan and Mr. S. Selva Nambi for encouraging my higher studies whole-heartedly.

I thank all my friends for their never ending love and support.

I thank my brothers who took away all the family burdens from me and for supporting my studies.

Last but not the least, I would like to thank my parents for the continued support they had given throughout my educational career and my life.

J. Raja

ABSTRACT

In this thesis, we first derive the lower dimensional model of linear elastic shallow shells using gamma convergence. We then justify the scalings used to derive the two dimensional model and finally we derive the two dimensional approximation of thin piezoelectric shallow shells with variable thickness.

In chapter 2, we consider the case of linear elastic shallow shells. Here we consider a boundary value problem in three dimensional elasticity posed over a shell of thickness 2ϵ having a specific geometry and clamped on a portion of its lateral surface. We then transfer the problem to a domain independent of the thickness parameter by suitable scalings on the unknowns and data and we show that the energy functionals $J(\epsilon)$ of the three-dimensional problem gamma converges to the energy functional associated with the two-dimensional problem and hence the sequence of functions which minimizes the energy associated with the three dimensional problem converge weakly to the function which minimizes the energy associated with a two dimensional model.

In chapter 3, we justify the scalings on the unknowns and data used to derive the two dimensional model of linearly elastic shallow shells.

The method of asymptotic analysis for deriving the two-dimensional models of plates and shells rely in a crucial way on appropriate scalings of the components of the displacement and appropriate assumptions on the data (Lamé constants and applied forces). The question is "are these scalings unique"?

This leads to the question of justifying the scalings used to derive these lower dimensional models.

We apply the formal asymptotic method to the variational formulation of the three-dimensional boundary value problems of linear shallow shells. Without making any *a priori* assumption of a mechanical or geometrical nature, we provide a complete justification of the scalings and assumptions.

In chapter 4, we consider thin piezoelectric shells with *variable* thickness. We first pose the problem in variational form and transfer the problem, by making suitable scalings on the unknowns and data, to a domain which is independent of the thickness parameter. We then show that the scaled solutions converge to a solution of a two dimensional model.

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NOTATIONS

Ω	domain in $I\!\!R^3$ (open, bounded, connected subset with a Lipschitz-continuous boundary, the set Ω being "locally on one side of its boundary").
$x = (x_i)$	generic point in $\overline{\Omega}$.
dx	volume element in Ω.
Γ	boundary of Ω.
(n_i)	unit normal vector along Γ.
$\Phi : \overline{\Omega} \subset I\!\!R^3 \longrightarrow I\!\!R^3$	injective and smooth enough mapping such that the three vectors $\partial_i \Phi$ are linearly independent at each point $x \in \overline{\Omega}$.
$g_i = \partial_i \Phi$	vectors of the covariant bases in the set $\Phi(\overline{\Omega})$.
g^i	vectors of the contravariant bases in the set $\Phi(\overline{\Omega})$. The vectors are defined at each $x \in \bar{\Omega}$ by the relations $g^i(x) \cdot g_j(x) = \delta^i_j$.
$g_{ij} = g^i \cdot g_j$	covariant components of the metric tensor of the $\Phi(\overline{\Omega})$.
$g = det(g_{ij})$	
$\Gamma^p_{ij} = g^p \cdot \partial_j g_i$	Christoffel symbols.
$v_{i\|\|j} = \partial_j v_i - \Gamma^p_{ij} v_p$	covariant derivatives of a vector field $v_i g^i$ with covariant components $v_i : \overline{\Omega} \longrightarrow I\!\!R$.
ω	domain in $I\!\!R^2$ (open, bounded, connected subset with a Lipschitz-continuous boundary, the set ω being "locally on one side of its boundary").
γ or $\partial\omega$	boundary of the set ω.
$d\gamma$	length element along γ .
γ_0	measurable subset of γ with length $\gamma_0 > 0$.
$x' = (x_\alpha)$	generic point in the set ω, sometimes also denoted y.

$\partial_\alpha = \frac{\partial}{\partial x_\alpha}, \quad \partial_{\alpha\beta} = \frac{\partial^2}{\partial x_\alpha x_\beta}$

$\Omega = \omega \times (-1, 1)$.

$(n_i) : \partial\Omega \to I\!R^3$ unit outer normal vector along the boundary $\partial\Omega$ of Ω.

$d\Gamma$ area element along $\partial\Omega$.

$\gamma \times [-\epsilon, \epsilon]$ lateral face of the set $\bar{\Omega}^\epsilon$.

$\Gamma_0^\epsilon = \gamma_0 \times [-\epsilon, \epsilon]$ portion of the lateral face where a shell is clamped.

$\Gamma_+^\epsilon = \gamma_0 \times \epsilon$ upper face of the set $\bar{\Omega}^\epsilon$.

$\Gamma_-^\epsilon = \gamma_0 \times -\epsilon$ lower face of the set $\bar{\Omega}^\epsilon$.

$\Delta = \partial_{\alpha\alpha}$ Laplacian.

$A^{ijkl} = \lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk})$.

contravariant components of the three- dimensional elasticity tensor.

$\hat{P}^{ijk,\epsilon}$ denote the piezoelectric tensors.

$\hat{\in}^{ij,\epsilon}$ denote the dielectric tensors.

$\mathcal{D}(\Omega)$ the space of functions in $\mathcal{C}^\infty(\Omega)$ with compact support in Ω.

$H^1(\Omega) = \{v \in L^2(\Omega);\ \partial_i v \in L^2(\Omega)\}$.

$H_0^1(\Omega) = \{v \in L^2(\Omega);\ v = 0 \text{ on } \partial\Omega\}$.

$H_\Gamma^1(\Omega) = \{v \in L^2(\Omega);\ v = 0 \text{ on } \partial\Gamma\}$.

GENERAL CONVENTIONS

1. Latin indices and exponents: $i, j, p, \ldots,$ take their values in the set {1, 2, 3}, unless otherwise indicated, as when they are used for indexing sequences.

2. Greek indices and exponents: $\alpha, \beta, \sigma, \ldots$ except ε, take their values in the set {1,2}.

3. The symbol "ϵ" designates a parameter that is > 0 and approaches zero.

CHAPTER 1

Introduction

Two dimensional model of plates and shells and one dimensional models of rods that rely on *a priori* assumptions of a mechanical and geometrical nature have been proposed by T. von Kármán, W. T. Koiter, L. Euler, P. M. Naghdi and others as approximation of the true three dimensional models when the thickness of the plate or shell or rod is "very small". The main reasons for preferring lower dimensional models are their amenability to numerical computations and their simpler mathematical structure produces richer variety of results.

Lower dimensional models being widely used, two essential questions arise. Given a "lower-dimensional" elastic body, together with specific loadings and boundary conditions, how to choose between the many lower-dimensional models that are available? For instance, given a linearly elastic shell, which theory should be preferred, among those of W. T. Koiter, M. Naghdi, T. von Kármán, L. Euler, etc? This question is of paramount practical importance, for it makes no sense to devise accurate methods for approximating the solution of a "wrong" model! Consequently, before approximating the exact solution of a given lower-dimensional model, we should first know whether it is "close enough" to the exact solution of the three-dimensional model it is intended to approximate. This observation leads to the second question: How to mathematically justify a lower-dimensional model from the three-dimensional model?

The first approach consists in directly estimating the difference between the three-dimensional solution and the solution of a given, i.e., "known in advance", lower-dimensional model. For linearly elastic plates, the first such estimate seems to be due to Morgenstern (1959). This approach was likewise successfully applied to linearly elastic shells by Koiter (1970a), Koiter (1970b) and Simmonds (1971).

The second approach, essentially due to Naghdi (1972) for plates and shells, consists in using the constraint method, whose governing principle is an *a priori* assumption that the admissible displacement fields are restricted to a specific form. References to this approach are Naghdi (1972) and Destuynder (1980).

These two approaches nevertheless rely on some *a priori* assumptions of a mechanical or geometrical nature, intended to account for the "smallness" of a geometrical parameter and intended to be more effective as this parameter approaches zero. Hence the need arises to mathematically justify these *a priori* assumptions, together with the lower-dimensional theories they engender, directly from three dimensional elasticity.

There are many approaches to justify the lower dimensional models. One way of doing is by formal asymptotic method. In this method, the three dimensional solution is first scaled in an appropriate manner so as to be defined in a fixed domain, then expanded as a formal series expansion in terms of ϵ, which is half the thickness of the material. The formal series expansion of the scaled solution is then inserted into the three-dimensional problem, and sufficiently many factors of the successive powers of ϵ found in this fashion are equated to zero until the leading term of the expansion can be computed and hopefully, identified with the scaled solution of a known lower dimensional model.

Using this method two dimensional models of linear and nonlinear plates, von Kármán and Margurre von Kármán models were derived by Ciarlet (1990). Fox et al. (1993) have derived the nonlinear invariant plates and Rao (1994) has applied this method to derive nonlinear spherical shell, nonlinear membrane model was derived by Miara (1998) and flexural shells were derived by Lods and Miara (1998) .

Another approach is to justify using asymptotic analysis in which one shows that the three-dimensional scaled solution converge in some Hilbert space to the solution of the lower-dimensional model. The main idea here is to establish a Korn's type inequality, depending on the geometry of the surface and the order of the forces, which helps in getting a priori estimates for the unknowns. Using this approach, the boundary value problems for linear elastic plates and shells were justified by Busse et al. (1997), Busse (1997), Ciarlet (1997), Ciarlet (2000) and the corresponding eigenvalue problems were justified by Ciarlet and Kesavan (1981), Kesavan (1979a), Kesavan (1979b), Kesavan and Sabu (1999), Kesavan and Sabu (2000a), Kesavan and Sabu (2000b), Sabu

(2002), Sabu (2003). Homogenization of a class of nonlinear eigenvalue problems has been studied by Baffico et al. (2006). The boundary value problem for rods was considered by Le Dret (1995). The dynamic problem for flexural shells were studied by Xiao (2001), membrane shells were studied by Xiao (1998) and generalised membrane shells were studied by Ji (2003). Busse (1998) has studied the case of linearly elastic membrane and flexural shells with *variable* thickness and Sabu (2001) has considered the case of linearly elastic shallow shells with *variable* thickness. Error estimation between the three dimensional and two dimensional solutions were studied by Mardare (1998a), Mardare (1998b). Regularity result for linear membrane shell has been studied by Genevey (1997). The idea of asymptotic analysis has been extended to study the asymptotic behaviour of a fluid in thin layers by Bunoiu and Kesavan (2004).

Third approach to justify the lower dimensional model is through Γ-convergence. Here the main idea is to show that the sequence of energy functionals associated with the three dimensional models converges to the energy functional associated with the lower dimensional model. Using this, Acerbi et al. (1991) have studied the case of elastic string, Bourquin et al. (1992) have justified the two-dimensional model of elastic plates and Genevey (2000) has justified the two-dimensional model of elastic membrane and flexural shells, Le Dret and Raoult (1995) have derived nonlinear membrane model, Le Dret and Raoult (1996) have justified the nonlinear membrane shell model. Mora et al. (2007), Mora and Müller (2008) have studied the case of elastic beams and Friesecke et al. (2006) have studied the nonlinear elastic plates and Müller and Pakzad (2008) have studied the von Kármán plates. Sabu (2010) has justified an one dimensional model of elastic rods. Homogenization of second order energies on periodic thin structures has been studied by Bouchitté et al. (2004).

Piezoelectricity is an electromechanical phenomenon. In general, a piezoelectric material responds to mechanical forces and generates an electric charge. Conversely, an electric charge applied to the material induces mechanical stress or strains. Piezoelectric materials are used as sensors and actuators, reduction of vibrations and noise.

Piezoelectric materials are also used in shape controlling for plane propellers, plane wings as well as in manufacturing artificial organs in biomechanics. A recent application of piezoelectric ultrasound sources is piezoelectric surgery, also known as piezosurgery.

In the recent past, there has been a phenomenal increase in the development of fiber-reinforced materials due to the desirability of lightweight, high strength and high-temperature performance requirements in modern technology. As fundamental structural elements, plates and shells of various geometries are widely used in various engineering fields such as, aerospace technology, missile technology etc. The appropriate variation of plate thickness or shell thickness provide the advantage of reduction in weight and size, and also have significantly greater efficiency for vibrations as compared to the plate of uniform thickness.

Theory of piezoelectric material is well developed by Banks et al. (1996), Ikeda (1990), Rahmoune et al. (1998), Tzou (1993) and lots of experimental works has been done in this area. However when the thickness of the material is very small, the behaviour of the piezoelectric material requires rigorous mathematical justification of the various models used - in analogy to the case of purely elastic materials. In this connection the boundary value problems for piezoelectric plates is studied by Sene (2001) and shells is studied by Bernadou and Haenel (2002) and Collard and Miara (2002). The corresponding eigenvalue problems are studied by Sabu (2002), Sabu (2003). Boundary value problem for piezoelectric membrane and flexural shells with *variable* thickness were studied by Sabu (2007).

Finite element method for various models in elasticity has been studied by Braess et al. (2007), Carstensen et al. (2012), Carstensen and Rabus (2012), Chilton and Suri (2000), Dauge and Suri (2002), Pitkäranta and Suri (2000) and many others.

CHAPTER 2

Justification of Two Dimensional Model of Shallow Shells using Gamma Convergence

2.1 Introduction

In recent year, a lot of works has been done on the mathematical justification of various classical lower-dimensional models for the study of thin linearly elastic shells.

There are many approaches to justify the lower dimensional models. One way of doing is through Γ-convergence, a powerful theory initiated by De Giorgi (1975), De Giorgi (1977). Here the main idea is to show that the sequence of energy functionals associated with the three dimensional models converges to the energy functional associated with the lower dimensional model.

Busse et al. (1997) have justified the two-dimensional model of elastic shallow shells using asymptotic analysis.

In this chapter, we justify this model using Γ convergence. We first show that the scaled energy functional $J(\varepsilon)$ associated with the three dimensional problem is weakly lower semi continuous. Then we construct a class of test functions for which the energy functional $J(\varepsilon)$ converges to the energy functional $J(v)$ of the two dimensional problem and then we show the strong convergence of the displacements.

2.2 The Three-Dimensional Problem

Throughout this thesis, Latin indices vary over the set $\{1, 2, 3\}$ and Greek indices over the set $\{1, 2\}$ for the components of vectors and tensors. The summation over repeated indices will be used.

Let $\omega \subset I\!R^2$ be a bounded domain with a Lipschitz continuous boundary γ and let ω lie locally on one side of γ. Let $\gamma_0 \subset \partial\omega$ with meas$(\gamma_0) > 0$. Let $\gamma_1 = \partial\omega \backslash \gamma_0$. For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon), \;\; \Gamma^{\pm,\varepsilon} = \omega \times \{\pm\varepsilon\}, \;\; \Gamma_0^\varepsilon = \gamma_0 \times (-\varepsilon, \varepsilon), \;\; \Gamma_1^\varepsilon = \gamma_1 \times (-\varepsilon, \varepsilon).$$

Let $x^\varepsilon = (x_1, x_2, x_3^\varepsilon)$ be a generic point on $\bar{\Omega}^\varepsilon$ and let $\partial_\alpha = \partial_\alpha^\varepsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\varepsilon = \frac{\partial}{\partial x_3^\varepsilon}$.

We assume that for each ε, we are given a function $\theta^\varepsilon : \bar{\omega} \to I\!R$ of class $\mathcal{C}^3$. We then define the map $\phi^\varepsilon : \bar{\omega} \to I\!R^3$ by

$$\phi^\varepsilon(x_1, x_2) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) \text{ for all } (x_1, x_2) \in \bar{\omega}. \tag{2.2.1}$$

At each point of the surface $S^\varepsilon = \phi^\varepsilon(\bar{\omega})$, we define the normal vector

$$a^\varepsilon = (|\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2 + 1)^{-\frac{1}{2}}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1).$$

For each $\varepsilon > 0$, we define the mapping $\Phi^\varepsilon : \bar{\Omega}^\varepsilon \to I\!R^3$ by

$$\Phi^\varepsilon(x^\varepsilon) = \phi^\varepsilon(x_1, x_2) + x_3^\varepsilon a^\varepsilon(x_1, x_2) \text{ for all } x^\varepsilon \in \bar{\Omega}^\varepsilon. \tag{2.2.2}$$

It can be shown (cf. Ciarlet (2000)) that there exists an $\varepsilon_0 > 0$ such that the mapping $\Phi^\varepsilon : \bar{\Omega}^\varepsilon \to \Phi^\varepsilon(\bar{\Omega}^\varepsilon)$ is a $\mathcal{C}^1$ diffeomorphism for all $0 < \varepsilon \leq \varepsilon_0$. The set $\bar{\hat{\Omega}}^\varepsilon = \Phi^\varepsilon(\bar{\Omega}^\varepsilon)$ is the reference configuration of the shell.

We define vectors g_i^ε and $g^{i,\varepsilon}$ by the relations

$$g_i^\varepsilon = \partial_i^\varepsilon \Phi^\varepsilon \quad \text{and} \quad g^{j,\varepsilon} \cdot g_i^\varepsilon = \delta_i^j,$$

which form the covariant and contravariant basis respectively of the tangent plane of $\Phi^\varepsilon(\bar{\Omega}^\varepsilon)$ at $\Phi^\varepsilon(x^\varepsilon)$. The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\varepsilon = g_i^\varepsilon \cdot g_j^\varepsilon \quad \text{and} \quad g^{ij,\varepsilon} = g^{i,\varepsilon} \cdot g^{j,\varepsilon}.$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\varepsilon} = g^{p,\varepsilon} \cdot \partial_j^{\varepsilon} g_i^{\varepsilon}.$$

Note however that when the set Ω^{ε} is of the special form $\Omega^{\varepsilon} = \omega \times (-\varepsilon, \varepsilon)$ and the mapping Φ^{ε} is of the form (2.2.2), the following relations hold,

$$\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0.$$

The volume element is given by $\sqrt{g^{\varepsilon}} dx^{\varepsilon}$ where

$$g^{\varepsilon} = det(g_{ij}^{\varepsilon}).$$

It can be shown that for ε sufficiently small, there exist constants g_1 and g_2 such that

$$0 < g_1 \leq g^{\varepsilon} \leq g_2. \tag{2.2.3}$$

Let $A^{ijkl,\varepsilon}$ be the elastic tensor. We assume that the material of the shell is *homogeneous and isotropic*. Then the elasticity tensor is given by

$$A^{ijkl,\varepsilon} = \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}) \tag{2.2.4}$$

where λ and μ are the Lamè constant of the material.

This tensor satisfies the following coercive relations. There exists a constant $C > 0$ such that for all symmetric tensors (t_{ij}),

$$A^{ijkl,\varepsilon} t_{kl} t_{ij} \geq C \sum_{i,j=1}^{3} (t_{ij})^2. \tag{2.2.5}$$

Moreover we have the symmetries

$$A^{ijkl,\varepsilon} = A^{klij,\varepsilon} = A^{jikl,\varepsilon}.$$

We define the space

$$V^\varepsilon = \{v \in (H^1(\Omega^\varepsilon))^3, v|_{\Gamma_0^\varepsilon} = 0\}. \tag{2.2.6}$$

Then the variational form of the problem is to find $u^\varepsilon \in V^\varepsilon$ such that

$$a^\varepsilon(u^\varepsilon, v^\varepsilon) = l^\varepsilon(v^\varepsilon) \text{ for all } v^\varepsilon \in V^\varepsilon \tag{2.2.7}$$

where

$$a^\varepsilon(u^\varepsilon, v^\varepsilon) = \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(u^\varepsilon) e^\varepsilon_{i||j}(v^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon, \tag{2.2.8}$$

$$l^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f^\varepsilon \cdot v^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon, \tag{2.2.9}$$

$$e^\varepsilon_{i||j}(v^\varepsilon) = \frac{1}{2}\left(\frac{\partial v^\varepsilon_i}{\partial x^\varepsilon_j} + \frac{\partial v^\varepsilon_j}{\partial x^\varepsilon_i}\right) - \Gamma^{p,\varepsilon}_{ij} v^\varepsilon_p. \tag{2.2.10}$$

Also u^ε can be characterized as the minimizer of the following functional.

$$J^\varepsilon(u^\varepsilon) = \min_{v^\varepsilon \in V^\varepsilon} J^\varepsilon(v^\varepsilon) \tag{2.2.11}$$

where

$$J^\varepsilon(v^\varepsilon) = \frac{1}{2}\left\{\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(v^\varepsilon) e^\varepsilon_{i||j}(v^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon\right\} - \int_{\Omega^\varepsilon} f^\varepsilon \cdot v^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \;\; \forall\, v^\varepsilon \in V^\varepsilon. \tag{2.2.12}$$

2.3 The Scaled Problem

We now perform a change of variable so that the domain no longer depends on ε.

Let $\Omega = \omega \times (-1,\ 1)$. With $x = (x_1, x_2, x_3) \in \bar{\Omega}$, we associate $x^\varepsilon = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon$. Let

$$\Gamma_0 = \gamma_0 \times (-1, 1), \;\; \Gamma_1 = \gamma_1 \times (-1, 1), \;\; \Gamma^\pm = \omega \times \{\pm 1\}.$$

With the functions $\Gamma^{p,\varepsilon}, g^\varepsilon, A^{ijkl,\varepsilon}$ we associate the functions $\Gamma^p(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$:

$\bar{\Omega} \to I\!R$ defined by

$$\Gamma^p(\varepsilon)(x) := \Gamma^{p,\varepsilon}(x^\varepsilon), \;\; g(\varepsilon)(x) = g^\varepsilon(x^\varepsilon), \;\; A^{ijkl}(\varepsilon)(x) = A^{ijkl,\varepsilon}(x^\varepsilon). \tag{2.3.1}$$

Assumption: We assume that the shell is a shallow shell; i.e., there exists a function $\theta \in \mathcal{C}^3(\bar{\omega})$ such that

$$\phi^\varepsilon(x_1, x_2) = (x_1, x_2, \varepsilon\theta(x_1, x_2)), \text{ for all } (x_1, x_2) \in \bar{\omega}. \tag{2.3.2}$$

In this case, we make the following scalings on the unknowns.

$$u^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), \quad v_\alpha(x^\varepsilon) = \varepsilon^2 v_\alpha(x), \tag{2.3.3}$$

$$u^\varepsilon_3(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), \quad v_3(x^\varepsilon) = \varepsilon v_3(x). \tag{2.3.4}$$

With the applied body forces f^ε, we associate the function $f(\varepsilon)$ through the relation

$$f^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 f_\alpha(\varepsilon)(x), \quad f^\varepsilon_3(x^\varepsilon) = \varepsilon^3 f_3(\varepsilon). \tag{2.3.5}$$

With the tensors $e^\varepsilon_{i||j}$, we associate the tensors $e_{i||j}(\varepsilon)$ through the relation

$$e^\varepsilon_{i||j}(v^\varepsilon)(x^\varepsilon) = \varepsilon^2 e_{i||j}(\varepsilon; v)(x). \tag{2.3.6}$$

We define the space

$$V = \{v \in (H^1(\Omega))^3, v|_{\Gamma_0} = 0\}. \tag{2.3.7}$$

With the energy functional J^ε, we associate the energy functional $J(\varepsilon)$ as

$$J^\varepsilon(v^\varepsilon) = \varepsilon^4 J(\varepsilon)(v(\varepsilon)). \tag{2.3.8}$$

Then the scaled unknown $u(\varepsilon)$ satisfies

$$\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)(u(\varepsilon))e_{i||j}(\varepsilon)(v)\sqrt{g(\varepsilon)}dx = \int_\Omega f_i(\varepsilon)v_i\sqrt{g(\varepsilon)}dx \text{ for all } v \in V \tag{2.3.9}$$

and $u(\varepsilon)$ is the minimizer of the functional

$$J(\varepsilon)(v) = \frac{1}{2}\Big\{\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)(v)e_{i||j}(\varepsilon)(v)\sqrt{g(\varepsilon)}dx\Big\} - \int_\Omega f_i(\varepsilon)v_i(\varepsilon)\sqrt{g(\varepsilon)}dx \ \forall v \in V. \tag{2.3.10}$$

2.4 Technical Preliminaries

In the sequel, we denote by $C_1, C_2, ...C_n$ various constants whose values do not depend on ε but may depend on θ.

Lemma 2.4.1. *The functions $e_{i||j}(\varepsilon, v)$ defined in (2.3.6) are of the form*

$$e_{\alpha||\beta}(\varepsilon)(v) = \tilde{e}_{\alpha\beta}(v) + \varepsilon^2 e^\sharp_{\alpha||\beta}(\varepsilon)(v), \tag{2.4.1}$$

$$e_{\alpha||3}(\varepsilon)(v) = \frac{1}{\varepsilon}\{\tilde{e}_{\alpha 3}(v) + \varepsilon^2 e^\sharp_{\alpha||3}(\varepsilon)(v)\}, \tag{2.4.2}$$

$$e_{3||3}(\varepsilon)(v) = \frac{1}{\varepsilon^2}\tilde{e}_{33}(v), \tag{2.4.3}$$

where

$$\tilde{e}_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - v_3\partial_{\alpha\beta}\theta = e_{\alpha\beta}(v) - v_3\partial_{\alpha\beta}\theta, \tag{2.4.4}$$

$$\tilde{e}_{\alpha 3}(v) = \frac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha) = e_{\alpha 3}(v), \tag{2.4.5}$$

$$\tilde{e}_{33}(v) = \partial_3 v_3 = e_{33}(v), \tag{2.4.6}$$

and there exists constant C_1 such that

$$\sup_{0<\varepsilon\leq\varepsilon_0}\max_{\alpha,j}||e^\sharp_{\alpha,j}(\varepsilon)(v)||_{0,\Omega} \leq C_1||v||_{1,\Omega} \ \textit{ for all } \ v \in V. \tag{2.4.7}$$

Also there exist constants C_2, C_3 and C_4 such that

$$\sup_{0<\varepsilon\leq\varepsilon_0}\max_{x\in\bar{\Omega}}|g(x) - 1| \leq C_2\varepsilon^2, \tag{2.4.8}$$

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{x\in\bar{\Omega}} |A^{ijkl}(\varepsilon) - A^{ijkl}| \leq C_3\varepsilon^2, \tag{2.4.9}$$

where

$$A^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), \tag{2.4.10}$$

and

$$A^{ijkl}(\varepsilon)t_{kl}t_{ij} \geq C_4 t_{ij}t_{ij} \tag{2.4.11}$$

for $0 < \varepsilon \leq \varepsilon_0$, *for all* $x \in \bar{\Omega}$, *and for all symmetric tensors* (t_{ij}).

Proof. A simple computation using the assumption (2.3.2) shows that

$$g_\alpha(\varepsilon) = \begin{pmatrix} \delta_{\alpha 1} - \varepsilon^2 x_3\partial_{\alpha 1}\theta + O(\varepsilon^4) \\ \delta_{\alpha 2} - \varepsilon^2 x_3\partial_{\alpha 2}\theta + O(\varepsilon^4) \\ \varepsilon\partial_\alpha\theta + O(\varepsilon^3) \end{pmatrix}, \quad g_3(\varepsilon) = \begin{pmatrix} -\varepsilon\partial_1\theta + O(\varepsilon^3) \\ -\varepsilon\partial_2\theta + O(\varepsilon^3) \\ 1 + O(\varepsilon^2) \end{pmatrix} \tag{2.4.12}$$

$$g^\alpha(\varepsilon) = \begin{pmatrix} \delta_{\alpha 1} + O(\varepsilon^2) \\ \delta_{\alpha 2} + O(\varepsilon^2) \\ \varepsilon\partial_\alpha\theta + O(\varepsilon^2) \end{pmatrix}, \quad g^3(\varepsilon) = \begin{pmatrix} -\varepsilon\partial_1\theta + O(\varepsilon^3) \\ -\varepsilon\partial_2\theta + O(\varepsilon^3) \\ 1 + O(\varepsilon^2) \end{pmatrix}, \tag{2.4.13}$$

$$g_{\alpha\beta}(\varepsilon) = \delta_{\alpha\beta} + \varepsilon^2[\partial_\alpha\theta\partial_\beta\theta - 2x_3\partial_{\alpha\beta}\theta] + O(\varepsilon^4), \;\; g_{\alpha 3}(\varepsilon) = O(\varepsilon), \;\; g_{33}(\varepsilon) = 1 + O(\varepsilon^2), \tag{2.4.14}$$

$$\Gamma^\sigma_{\alpha\beta}(\varepsilon) = O(\varepsilon^2), \;\; \Gamma^3_{\alpha\beta}(\varepsilon) = \varepsilon\partial_{\alpha\beta}(\theta) + O(\varepsilon^3), \;\; \Gamma^\sigma_{\alpha 3}(\varepsilon) = O(\varepsilon). \tag{2.4.15}$$

The announced results follows from the above relations. □

The following lemma (cf. Busse et al. (1997)) plays an important role in the convergence analysis.

Lemma 2.4.2. *Let* $\theta \in \mathcal{C}^3(\bar{\omega})$ *be a given function and let the functions* $\tilde{e}_{ij}$ *be defined as in (2.4.4)-(2.4.6). Then there exists a constant* C_5 *such that the following generalised Korn's inequality holds.*

$$||v||_{1,\Omega} \leq C_5 \left\{ \sum_{i,j} ||\tilde{e}_{ij}(v)||^2_{0,\Omega} \right\}^{\frac{1}{2}} \tag{2.4.16}$$

for all $v \in V$ where V is the space defined in (2.3.7).

Definition 2.4.1. *Let V be a Banach space and $(J(\varepsilon))_{\varepsilon>0}$ a sequence of functionals $J(\varepsilon) : V \to I\!R \cup \{\infty\}$. We say that the functional $J : V \to I\!R$ is the $\Gamma-$limit of the functionals $J(\varepsilon)$ if the following properties holds.*

(i) If $(v(\varepsilon))_{\varepsilon>0} \rightharpoonup v$ in V implies $J(v) \leq \liminf_{\varepsilon\to 0} J(\varepsilon)(v(\varepsilon))$.

(ii) For every $v \in V$, there exists a sequence $(v(\varepsilon))_{\varepsilon>0} \in V$ such that

$$(v(\varepsilon))_{\varepsilon>0} \rightharpoonup v \text{ and } J(\varepsilon)(v(\varepsilon)) \to J(v).$$

Remark 2.4.1. *It can be shown that when the $\Gamma-$limit exists, it is unique.*

The main result from Γ-convergence is the following, see Dal Maso (1989).

Theorem 2.4.3. *Assume that the sequence $(J(\varepsilon))_{\varepsilon>0}$, Γ-converges to J, and assume that there exists a compact subset U of V independent of ε such that, for all $\varepsilon > 0$, there exists $u(\varepsilon)$ satisfying*

$$u(\varepsilon) \in U \quad \text{and} \quad J(\varepsilon)(u(\varepsilon)) = \inf_{v\in V} J(\varepsilon)(v).$$

Then there exists $u \in U$ such that

$$u(\varepsilon) \rightharpoonup u \quad \text{and} \quad J(u) = \inf_{v\in V} J(v).$$

Moreover, one has

$$J(\varepsilon)(u(\varepsilon)) \to J(u).$$

2.5 Convergence of the Scaled Solutions

Let V_{KL} be the space defined by

$$V_{KL} = \{v = (v_i) \in (H^1(\Omega))^3; e_{i3}(v) = 0 \text{ in } \Omega, \;\; v_i = 0 \;\text{ on } \Gamma_0\}. \tag{2.5.1}$$

For any $v \in V$, define

$$J(v) = \begin{cases} \frac{1}{2}\int_\Omega \left\{ \frac{2\lambda\mu}{\lambda+2\mu}\tilde{e}_{\sigma\sigma}(v)\tilde{e}_{\tau\tau}(v) + 2\mu\tilde{e}_{\alpha\beta}(v)\tilde{e}_{\alpha\beta}(v)\right\} dx - \int_\Omega f_i v_i dx & \text{if } v \in V_{KL}, \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 2.5.1. *The functional J is the $\Gamma-$ limit of the functional $J(\varepsilon)$ for the weak topology of the space V.*

Proof. Note that the functional $J(\varepsilon)$ can be written as

$$\begin{aligned} J(\varepsilon)(v) = & \frac{1}{2}\int_\Omega \frac{2\lambda\mu}{\lambda+2\mu} g^{\alpha\beta}(\varepsilon) g^{\sigma\tau}(\varepsilon) e_{\alpha||\beta}(\varepsilon)(v) e_{\sigma||\tau}(\varepsilon)(v)\sqrt{g(\varepsilon)}dx \\ & + \frac{1}{2}\int_\Omega 2\mu g^{\alpha\sigma}(\varepsilon) g^{\beta\tau}(\varepsilon) e_{\alpha||\beta}(\varepsilon)(v) e_{\sigma||\tau}(\varepsilon)(v)\sqrt{g(\varepsilon)}dx \\ & + \frac{1}{2}\int_\Omega \left\{ (\lambda+2\mu)\left[\frac{\lambda}{\lambda+2\mu} g^{\alpha\beta}(\varepsilon) e_{\alpha||\beta}(\varepsilon)(v) + g^{33}(\varepsilon) e_{3||3}(\varepsilon)(v)\right]^2 \right\} \sqrt{g(\varepsilon)}dx \\ & + \frac{1}{2}\int_\Omega 4\mu g^{\alpha\sigma}(\varepsilon) g^{33}(\varepsilon) e_{\alpha||3}(\varepsilon)(v) e_{\sigma||3}(\varepsilon)(v)\sqrt{g(\varepsilon)}dx \\ & - \int_\Omega f_i(\varepsilon) v_i \sqrt{g(\varepsilon)}dx. \end{aligned} \tag{2.5.2}$$

Step 1: We first show that

$$v(\varepsilon) \rightharpoonup v \ \ in \ \ V \Rightarrow J(v) \leq \liminf_{\varepsilon\to 0} J(\varepsilon)(v(\varepsilon)). \tag{2.5.3}$$

If $v \notin V_{KL}$, then from the definition of J, it follows that $J(v) = \infty$. Hence it is enough to show that

$$\liminf_{\varepsilon\to 0} J(\varepsilon)(v(\varepsilon)) = \infty.$$

Suppose that

$$\liminf_{\varepsilon\to 0} J(\varepsilon)(v(\varepsilon)) < \infty.$$

Then it follows that there exists a constant $C_6 > 0$ and a subsequence $(v(\varepsilon))_{\varepsilon>0}$ (still denoted by ε) such that

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon)(v(\varepsilon)) e_{i||j}(\varepsilon)(v(\varepsilon))\sqrt{g(\varepsilon)}dx \leq C_6. \tag{2.5.4}$$

Hence from the relations (2.2.5), (2.4.8)-(2.4.11) and the Lemma 2.4.2, it follows that

$$||e_{i||j}(\varepsilon)(v(\varepsilon)||_{0,\Omega} \leq C_7, \quad ||v_i(\varepsilon)||_{1,\Omega} \leq C_7.$$

Hence there exists subsequence $(e_{i||j}(\varepsilon)v(\varepsilon))_{\varepsilon>0}$ and functions $e_{i||j} \in L^2(\Omega)$ and $v \in H^1(\Omega)$ such that

$$e_{i||j}(\varepsilon)(v(\varepsilon)) \rightharpoonup e_{i||j} \text{ weakly in } L^2(\Omega), \tag{2.5.5}$$

$$v_i(\varepsilon) \rightharpoonup v_i \text{ weakly in } H^1(\Omega). \tag{2.5.6}$$

Using the convergences (2.5.5)-(2.5.6), it is a standard argument (cf. Busse et al. (1997)) to show that $e_{i3}(v) = 0$, and hence $v \in V_{KL}$ which is a contradiction. Hence

$$\liminf_{\varepsilon \to 0} J(\varepsilon)(v(\varepsilon)) = \infty.$$

Assume next that $v \in V_{KL}$. Suppose

$$\liminf_{\varepsilon \to 0} J(\varepsilon)(v(\varepsilon)) = \infty$$

then (2.5.3) always holds. Suppose that

$$\liminf_{\varepsilon \to 0} J(\varepsilon)(v(\varepsilon)) < \infty.$$

As in the first case, there exist subsequence $(v(\varepsilon))_{\varepsilon>0}$ such that the convergences (2.5.5)-(2.5.6) holds and since $v(\varepsilon) \rightharpoonup v$ in $H^1(\Omega)$, it follows from the definition that $e_{\alpha||\beta}(\varepsilon)(v(\varepsilon)) \rightharpoonup \tilde{e}_{\alpha\beta}(v)$ in $L^2(\Omega)$.

From the positive definiteness of $A^{ijkl}(\varepsilon)$ and (2.5.2) it follows that

$$\begin{aligned} J(\varepsilon)(v) \geq & \frac{1}{2}\int_\Omega \frac{2\lambda\mu}{\lambda+2\mu} g^{\alpha\beta}(\varepsilon) g^{\sigma\tau}(\varepsilon) e_{\alpha||\beta}(\varepsilon)(v) e_{\sigma||\tau}(\varepsilon)(v)\sqrt{g(\varepsilon)}dx \\ & + \frac{1}{2}\int_\Omega 2\mu g^{\alpha\sigma}(\varepsilon) g^{\beta\tau}(\varepsilon) e_{\alpha||\beta}(\varepsilon)(v) e_{\sigma||\tau}(\varepsilon)(v)\sqrt{g(\varepsilon)}dx \\ & - \int_\Omega f_i(\varepsilon) v_i(\varepsilon)\sqrt{g(\varepsilon)}dx. \end{aligned} \tag{2.5.7}$$

With the convergence $g^{ij}(\varepsilon) \to \delta_{ij},\ \ g(\varepsilon) \to 1$ in $\mathcal{C}(\Omega)$ and the weak convergence $e_{\alpha||\beta}(\varepsilon)(v(\varepsilon)) \rightharpoonup \tilde{e}_{\alpha\beta}(v)$, it follows that for any convergent subsequence $J(\varepsilon)(v(\varepsilon))$ we have

$$\begin{aligned}\lim_{\varepsilon\to 0} J(\varepsilon)(v(\varepsilon)) &\geq \frac{1}{2}\int_\Omega \left\{\frac{2\lambda\mu}{\lambda+2\mu}\tilde{e}_{\sigma\sigma}(v)\tilde{e}_{\tau\tau}(v) + 2\mu\tilde{e}_{\alpha\beta}(v)\tilde{e}_{\alpha\beta}(v)\right\} dx - \int_\Omega f_i v_i dx \\ &= J(v). \qquad (2.5.8)\end{aligned}$$

Hence (2.5.3) follows.

Step 2: We show that for any $v \in V$, there exists a sequence $(v(\varepsilon))_{\varepsilon>0}$ such that

$$v(\varepsilon) \rightharpoonup v \ \text{ in } \ V \ \text{ and } \ J(v) = \lim_{\varepsilon\to 0} J(\varepsilon)(v(\varepsilon)). \qquad (2.5.9)$$

If $v \notin V_{KL}$, then by taking $v(\varepsilon) = v$, it follows from step 1 and the definition of the functional J that

$$J(v) = \liminf_{\varepsilon\to 0} J(\varepsilon)(v(\varepsilon)) = \infty \qquad (2.5.10)$$

and hence the property (2.5.9) holds.

Define the space

$$W = \left\{(\eta_\alpha - x_3\partial_\alpha\eta_3,\ \eta_3),\ \eta_\alpha \in H^2(\omega), \eta_3 \in H^2(\omega), \eta_i = \partial_\nu\eta_3 = 0,\ \ on\ \ \gamma_0\right\}. \qquad (2.5.11)$$

Let $v \in W$. Define $v(\varepsilon) \in V$ by

$$v_\alpha(\varepsilon) = v_\alpha,\quad v_3(\varepsilon) = \eta_3 - \varepsilon^2\frac{\lambda}{\lambda+2\mu}\left(x_3(\partial_\sigma\eta_\sigma - \eta_3\partial_{\alpha\alpha}\theta) - \frac{x_3^2}{2}\Delta\eta_3\right) \qquad (2.5.12)$$

Then as $\varepsilon \to 0$, we have

$$v(\varepsilon) \to v, \qquad (2.5.13)$$

$$e_{\alpha||3}(v(\varepsilon)) = \varepsilon\frac{\lambda}{\lambda+2\mu}\partial_\alpha\left(x_3(\partial_\sigma\eta_\sigma - \eta_3\partial_{\alpha\alpha}\theta) - \frac{x_3^2}{2}\Delta\eta_3\right) \to 0, \qquad (2.5.14)$$

$$\frac{\lambda}{\lambda+2\mu}e_{\sigma||\sigma}(v(\varepsilon)) + e_{3||3}(v(\varepsilon)) = \varepsilon^2\left(\frac{\lambda}{\lambda+2\mu}\right)^2\left(x_3(\partial_\sigma\eta_\sigma - \eta_3\partial_{\alpha\alpha}\theta) - \frac{x_3^2}{2}\Delta\eta_3\right) \to 0, \qquad (2.5.15)$$

Using the above convergences and relations (2.4.14) in (2.5.2) it follows that

$$J(\varepsilon)(v(\varepsilon)) \to J(v).$$

Since the space W is dense in the space V_{KL} the above convergence hold for any $v \in V_{KL}$. $\square$

Theorem 2.5.2. *For each $\varepsilon > 0$, let $(u(\varepsilon))$ be the minimizer of the functional $J(\varepsilon)(v)$ defined by (2.3.10). Then*

$$u_i(\varepsilon) \to u \ \ in \ \ H^1(\Omega), \quad u \in V_{KL}, \tag{2.5.16}$$

and u is the solution of the minimization problem

$$J(u) = \min_{v \in V} J(v). \tag{2.5.17}$$

Proof. It follows from the inequality (2.4.16) that $||u(\varepsilon)||_{1,\Omega}$ are bounded independent of ε. Thus $\{u(\varepsilon)\}_{\varepsilon>0}$ belong to a weakly compact subset of V. Moreover the weak limit u of $u(\varepsilon)$ belongs to V_{KL} (cf.Busse et al. (1997)). Then it follows from the above theorem that there exists a subsequence $u(\varepsilon_k)$ such that $u(\varepsilon_k) \rightharpoonup u$ in V and u satisfies

$$J(u) = \inf_{v \in V} J(v).$$

Thus the function is unique and the whole sequence $u(\varepsilon)$ converges weakly to u in V.

To show that the family $u(\varepsilon)$ converges strongly to u in $H^1(\Omega)$, it is enough to show by virtue of (2.4.16) that

$$\tilde{e}_{ij}(u(\varepsilon)) \to \tilde{e}_{ij}(u) \ \ in \ \ L^2(\Omega). \tag{2.5.18}$$

Define

$$K_{\alpha\beta}(\varepsilon) = \tilde{e}_{\alpha\beta}(u(\varepsilon)), \ \ K_{\alpha 3}(\varepsilon) = \frac{1}{\varepsilon}\tilde{e}_{\alpha 3}(u(\varepsilon)), \ \ K_{33}(\varepsilon) = \frac{1}{\varepsilon^2}\tilde{e}_{33}(u(\varepsilon)) \tag{2.5.19}$$

and

$$K_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u), \;\; K_{\alpha 3} = 0, \;\; K_{33} = -\frac{\lambda}{\lambda + 2\mu}\tilde{e}_{\alpha\alpha}(u). \tag{2.5.20}$$

Claim: $K(\varepsilon) = (K_{ij}(\varepsilon)) \rightharpoonup K = (K_{ij})$ weakly in $L^2(\Omega)$.

It follows from (2.2.5) and (2.3.9) that

$$C_8 \sum_{i,j} ||e_{i||j}(\varepsilon; u(\varepsilon))||^2_{0,\Omega} \leq \int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon; u(\varepsilon)) e_{i||j}(\varepsilon; u(\varepsilon)) \sqrt{g(\varepsilon)} dx$$
$$\leq ||f||_{0,\Omega} ||u(\varepsilon)||_{0,\Omega}. \tag{2.5.21}$$

Hence $(e_{i||j}(\varepsilon, u(\varepsilon)))$ is bounded.

From the definition (2.5.20) and relations (2.4.1)-(2.4.6), we have

$$||K(\varepsilon)||^2_{0,\Omega} \leq 2 \sum_{i,j} ||e_{i||j}(\varepsilon; u(\varepsilon))||^2_{0,\Omega} + 2\varepsilon^4 \sum_{\alpha\beta} ||\tilde{e}^\sharp(\varepsilon; u(\varepsilon))||^2_{0,\Omega} + 4\varepsilon^2 \sum_{\alpha} ||\tilde{e}^\sharp(\varepsilon; u(\varepsilon))||^2_{0,\Omega}. \tag{2.5.22}$$

From the boundedness of $(e_{i||j}(\varepsilon, u(\varepsilon)))$ and the relation (2.4.7) it follows that $(K(\varepsilon))$ is bounded and hence $K(\varepsilon) \rightharpoonup K$ in $(L^2(\Omega))^9$ weakly.

Clearly $K_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u)$.

We next note the following result (cf.Ciarlet (1990)).

$$\int_\Omega u \partial_3 v dx = 0 \;\text{ for all }\; v \in H^1(\Omega) \;\text{ with }\; v = 0 \;\text{ on }\; \Gamma_0 \;\Rightarrow u = 0. \tag{2.5.23}$$

Multiplying (2.3.9) by ε and taking $v_3 = 0$ we get

$$2 \int_\Omega A^{\alpha 3 \sigma 3}(0) K_{\alpha 3}(\varepsilon) \partial_3 v_\alpha dx = \varepsilon R(\varepsilon, K(\varepsilon), u(\varepsilon), v) \tag{2.5.24}$$

where $R(\varepsilon, K(\varepsilon), u(\varepsilon), v)$ is bounded independent of ε. Passing to the limit as $\varepsilon \to 0$ in (2.5.24) we get

$$\int_\Omega K_{\alpha 3} \partial_3 v_\alpha dx = 0 \;\text{ for all }\; v_\alpha. \tag{2.5.25}$$

Hence $K_{\alpha 3} = 0$. Multiplying (2.3.9) by ε^2 and letting $v_\alpha = 0$ we get

$$\int_\Omega \{A^{33\sigma\tau}(0)K_{\sigma\tau}(\varepsilon) + A^{3333}(0)K_{33}(\varepsilon)\}\partial_3 v_3 dx = \int_\Omega \{\lambda K_{\sigma\sigma}(\varepsilon) + (\lambda + 2\mu)K_{33}(\varepsilon)\}\partial_3 v_3 dx$$
$$= \varepsilon S(\varepsilon, K(\varepsilon), u(\varepsilon), v), \qquad (2.5.26)$$

where $S(\varepsilon, K(\varepsilon), u(\varepsilon), v)$ is independent of ε. Letting $\varepsilon \to 0$, we get

$$\int_\Omega \{\lambda K_{\sigma\sigma} + (\lambda + 2\mu)K_{33}\} \, \partial_3 v_3 \, dx = 0. \qquad (2.5.27)$$

Hence $K_{33} = -\frac{\lambda}{\lambda+2\mu}\tilde{e}_{\sigma\sigma}(u)$. Since $\tilde{e}_{i3}(u) = 0$ and

$$\sum_{i,j} ||\tilde{e}_{ij}(u(\varepsilon)) - \tilde{e}_{ij}(u)||^2_{0,\Omega}$$
$$= \sum_{\alpha,\beta} ||K_{\alpha\beta}(\varepsilon) - K_{\alpha\beta}||^2_{0,\Omega} + 2\varepsilon^2 \sum_\alpha ||K_{\alpha 3}(\varepsilon)||^2_{0,\Omega} + \varepsilon^2 ||K_{33}(\varepsilon)||^2_{0,\Omega} \qquad (2.5.28)$$

the convergence (2.5.18) is equivalent to showing that $K(\varepsilon) \to K$ strongly in $L^2(\Omega)$.

For any two symmetric matrices $S = (s_{ij})$ and $T = (t_{ij})$, define

$$AS : T = A^{ijkl}(0)t_{kl}t_{ij} = \lambda s_{pp}t_{qq} + 2\mu s_{ij}t_{ij}.$$

Then

$$\begin{aligned} \int_\Omega AK : K dx &= \int_\Omega \{\lambda K_{pp}K_{qq} + 2\mu K_{ij}K_{ij}\} dx \\ &= \int_\Omega \left\{ \frac{2\lambda\mu}{\lambda + 2\mu} \tilde{e}_{\sigma\sigma}(u)\tilde{e}_{\tau\tau}(u) + 2\mu \tilde{e}_{\alpha\beta}(u)\tilde{e}_{\alpha\beta}(u) \right\} dx \\ &= \int_\Omega f_i u_i dx. \end{aligned} \qquad (2.5.29)$$

Taking $v = u(\varepsilon)$ in (2.3.9), and using the relations (2.4.1) - (2.4.10), we get

$$\int_\Omega AK(\varepsilon) : K(\varepsilon) dx = \int_\Omega f_i u_i(\varepsilon) dx + \varepsilon r(\varepsilon, u(\varepsilon)), \qquad (2.5.30)$$

where there exists a constant C_9 such that

$$\sup_{0<\varepsilon\leq e_1} |r(\varepsilon, u(\varepsilon))| \leq C_9. \tag{2.5.31}$$

From the relations (2.5.30) - (2.5.31) and the weak convergence of $u(\varepsilon)$ we deduce that

$$\int_\Omega AK(\varepsilon) : K(\varepsilon) dx \to \int_\Omega f_i u_i dx \ \ as \ \ \varepsilon \to 0. \tag{2.5.32}$$

Also, using (2.5.29) - (2.5.32) it follows that

$$\begin{aligned} 2\mu\|K(\varepsilon) - K\|_{0,\Omega}^2 &\leq \int_\Omega A(K(\varepsilon) - K) : (K(\varepsilon) - K) dx \\ &= \int_\Omega AK(\varepsilon) : K(\varepsilon) dx + \int_\Omega AK : (K - 2K(\varepsilon)) dx \\ &\to \int_\Omega f_i u_i dx - \int_\Omega AK : K dx \\ &= 0. \end{aligned} \tag{2.5.33}$$

Hence the convergence (2.5.18) follows. □

Theorem 2.5.3. *Let u be the minimizer of the functional J. Let*

$$V_H(\omega) = \{(\eta_\alpha) \in (H^1(\omega))^2 : \eta_\alpha = 0 \ \ on \ \ \gamma_0\} \tag{2.5.34}$$

$$V_3(\omega) = \{\eta_3 \in H^2(\omega) : \eta_3 = \partial_\nu \eta_3 = 0 \ \ on \ \ \gamma_0\} \tag{2.5.35}$$

Then there exists $(\zeta_i) \in V_H(\omega) \times V_3(\omega)$ such that

(a) $u_\alpha(x) = \zeta_\alpha - x_3 \partial_\alpha \eta_3, \quad u_3(x) = \zeta_3(x_3)$

(b) (ζ_i) *solves the following variational equations:*

$$-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega - \int_\omega n_{\alpha\beta}\partial_{\alpha\beta}\theta\eta_3 d\omega = \int_\omega p_3\eta_3 d\omega - \int_\omega q_\alpha \partial_\alpha \eta_3 d\omega \ \ \forall \ \ \eta_3 \in V_3(\omega), \tag{2.5.36}$$

$$\int_\omega n_{\alpha\beta}\partial_\beta \eta_\alpha d\omega = \int_\omega p_\alpha \eta_\alpha d\omega \ \ \forall \ \ \eta_\alpha \in V_H(\omega). \tag{2.5.37}$$

where

$$m_{\alpha\beta} = -\left\{\frac{4\lambda\mu}{3(\lambda+2\mu)} \triangle \zeta_3\delta_{\alpha\beta} + \frac{4}{3}\mu\partial_{\alpha\beta}\zeta_3\right\}, \tag{2.5.38}$$

$$n_{\alpha\beta} = \left\{\frac{4\lambda\mu}{(\lambda+2\mu)}\tilde{e}_{\rho\rho}(\zeta)\delta_{\alpha\beta} + 4\mu\tilde{e}_{\alpha\beta}(\zeta)\right\}, \tag{2.5.39}$$

$$\tilde{e}_{\alpha\beta} = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) - \zeta_3\partial_{\alpha\beta}\theta, \tag{2.5.40}$$

$$p_i = \int_{-1}^{1} f_i dx_3 \;\; q_\alpha = \int_{-1}^{1} x_\alpha f_3 dx_3. \tag{2.5.41}$$

Proof. Since u is the minimizer of the functional J, $u \in V_{KL}$ and satisfies

$$\frac{1}{2}\int_\Omega \left\{\frac{2\lambda\mu}{\lambda+\mu}\tilde{e}_{\sigma\sigma}(u)\tilde{e}_{\tau\tau}(w) + 2\mu\tilde{e}_{\alpha\beta}(u)\tilde{e}_{\alpha\beta}(w)\right\} dx = \int_\Omega f_i w_i dx \quad \forall w \in V_{KL}. \tag{2.5.42}$$

Equations (2.5.36) and (2.5.37) follow by taking $\eta_\alpha = 0$ and $\eta_3 = 0$ respectively in the above equation.

□

CHAPTER 3

Justification of Asymptotic Analysis of Linear Shallow Shells

3.1 Introduction

Two dimensional models are derived from three dimensional models under suitable scalings on the unknowns and data when the thickness of the material is very small. The question is "are these scalings unique"? This leads to the question of justifying the scalings used to derive these lower dimensional models. In this direction Miara (1994a), Miara (1994b) has justified the scalings used to derive the linear and nonlinear plate models respectively. The purpose of this work is to justify the scalings on the unknowns and data used to derive the two dimensional model of linearly elastic shallow shells.

The variational formulation of the three-dimensional problem in linearized elasticity is to find the displacement vector $u^\varepsilon \in V^\varepsilon$ such that

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(u^\varepsilon) e^\varepsilon_{i||j}(v^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon} dx^\varepsilon \quad \forall \ v^\varepsilon \in V^\varepsilon \tag{3.1.1}$$

Assumption: We assume that the shell is shallow shell, that is there exists $\theta \in \mathcal{C}^3(\bar{\omega})$ such that $\theta^\varepsilon(y) = \varepsilon\theta(y)$ for all $y \in \bar{\omega}$.

To study the behaviour of displacement field $u^\varepsilon = (u^\varepsilon_i) : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ as ε goes to zero, the following three basic ideas are used.

(i) The three dimensional problem whose solution u^ε is posed over the variable set $\bar{\Omega}^\varepsilon$, is transformed into a three-dimensional problem over the fixed set $\bar{\Omega}$ according to the correspondence :

$$x^\varepsilon = (x^\varepsilon_i) \in \bar{\Omega}^\varepsilon \leftrightarrow x = (x_i) \in \bar{\Omega} \ \text{ with } \ x_\alpha = x^\varepsilon_\alpha, \ \ x^\varepsilon_3 = \varepsilon x_3. \tag{3.1.2}$$

(ii) The components $(u_i^\varepsilon) : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ of the displacement field are scaled, in the sense that functions $u_i(\varepsilon) : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ are associated to the functions u_i^ε through the relations:

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), \;\; u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x) \;\text{ for all } x \in \Omega. \tag{3.1.3}$$

(iii) It is assumed that there exists constants $\lambda > 0$ and $\mu > 0$ independent of ε such that

$$\lambda^\varepsilon = \lambda, \quad \mu^\varepsilon = \mu \tag{3.1.4}$$

and that there exist functions $f = (f_i) : \bar{\Omega} \to \mathbb{R}^3$ independent of ε, such that

$$f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 f_\alpha(x), \quad f_3^\varepsilon(x^\varepsilon) = \varepsilon^3 f_3(x) \tag{3.1.5}$$

Under the above scalings, it was shown in Busse et al. (1997) (and also in the previous chapter) that $u(\varepsilon) \longrightarrow u$ in V, $u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3$, $u_3 = \zeta_3$ where $\zeta = (\zeta_i)$ is the solution of two dimensional shallow shell model

$$-\int_\omega m^{\alpha\beta}(\zeta)\partial_{\alpha\beta}\eta_3 d\omega - \int_\omega \widetilde{n}^{\alpha\beta}(\zeta)\eta_3\partial_{\alpha\beta}\theta d\omega + \int_\omega \widetilde{n}^{\alpha\beta}(\zeta)\partial_\beta\eta_\alpha d\omega = \int_\omega p^i\eta_i d\omega - \int_\omega q^i\partial_\alpha\eta_3 d\omega \tag{3.1.6}$$

for all $\eta = (\eta_i) \in V(\omega) = V_H(\omega) \times V_3(\omega)$.
where

$$V_H(\omega) = \{\eta = (\eta_\alpha) \in (H^1(\omega))^2;\ \eta_\alpha = 0 \text{ on } \gamma_0\}, \tag{3.1.7}$$

$$V_3(\omega) = \{\eta_3 \in H^2(\omega);\ \eta_3 = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}, \tag{3.1.8}$$

$$m^{\alpha\beta}(\zeta) = -\left\{\frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta\zeta_3\delta_{\alpha\beta} + \frac{4\mu}{3}\partial_{\alpha\beta}\zeta_3\right\}, \tag{3.1.9}$$

$$\widetilde{n}^{\alpha\beta}(\zeta) = \frac{4\lambda\mu}{(\lambda+2\mu)}\widetilde{e}_{\sigma\sigma}(\zeta)\delta_{\alpha\beta} + 4\mu\widetilde{e}_{\alpha\beta}(\zeta), \tag{3.1.10}$$

$$\widetilde{e}_{\alpha\beta}(\zeta) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) - \zeta_3\partial_{\alpha\beta}\theta, \tag{3.1.11}$$

$$p^\alpha = \int_{-1}^{1} f_\alpha^1 dx_3, \qquad p^3 = \int_{-1}^{1} f_3^2 dx_3, \qquad q^\alpha = \int_{-1}^{1} x_3 f_\alpha^1 dx_3. \tag{3.1.12}$$

Remark 3.1.1. *Since the problem is linear the following assumptions on the displace-*

ments and forces are also possible for any real number p.

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^{2+p} u_\alpha(\varepsilon)(x), \quad u_3^\varepsilon(x^\varepsilon) = \varepsilon^{1+p} u_3(\varepsilon)(x),$$
$$f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^{2+p} f_\alpha(x), \qquad f_3^\varepsilon(x^\varepsilon) = \varepsilon^{3+p} f_3(x).$$

The objective of this work is to give a mathematical justification of the *relative order* of scalings between the 'horizontal components' $u_\alpha^\varepsilon,\ f_\alpha^\varepsilon$ and the 'vertical components' $u_3^\varepsilon,\ f_3^\varepsilon$ by assuming only a formal asymptotic expansion of the unknown displacement field.

3.2 The Three-dimensional Problem in Scaled Domain

With the functions $A^{ijkl,\varepsilon},\ g^\varepsilon$ we associate the functions $A^{ijkl}(\varepsilon) : \bar{\Omega} \to \mathbb{R}$ and $g(\varepsilon) : \bar{\Omega} \to \mathbb{R}$ through the relations

$$A^{ijkl,\varepsilon}(x^\varepsilon) = A^{ijkl}(\varepsilon)(x), \ \ g^\varepsilon(x^\varepsilon) = g(\varepsilon)(x) \quad \text{for all } x \in \bar{\Omega}.$$

With the functions $v^\varepsilon \in V^\varepsilon$ we associate $v(\varepsilon) \in V$ through the relations

$$v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(\varepsilon)(x), \quad v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(\varepsilon)(x).$$

With the functions $e_{i||j}^\varepsilon(v^\varepsilon)$ we associate the functions $e_{i||j}(\varepsilon; v(\varepsilon))$ through the relation

$$e_{i||j}^\varepsilon(v^\varepsilon)(x^\varepsilon) = e_{i||j}(\varepsilon; v(\varepsilon))(x) \quad \text{for all } x \in \Omega.$$

Then the variational equation (3.1.1) posed on the fixed domain Ω is to find $u(\varepsilon) \in V$ such that

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon; u(\varepsilon)) e_{i||j}(\varepsilon; v) \sqrt{g(\varepsilon)} dx = \int_\Omega f^i v_i \sqrt{g(\varepsilon)} dx \ \ \forall v \in V. \qquad (3.2.1)$$

By Lemma 2.4.1 we get the following relations

$$\left.\begin{aligned}
g_{\alpha\beta} &= \delta_{\alpha\beta} + \varepsilon^2(\partial_\alpha\theta\cdot\partial_\beta\theta - 2x_3\partial_{\alpha\beta}\theta) + O(\varepsilon^4), \;\; g_{i3}(\varepsilon) = \delta_{i3} \\
\sqrt{g(\varepsilon)} &= 1 + \varepsilon^2 g^1(x_1, x_2, x_3) + O(\varepsilon^4) \\
\Gamma^\sigma_{\alpha\beta}(\varepsilon) &= 0 + \varepsilon^2\Gamma^{\sigma,2}_{\alpha\beta} + \varepsilon^3\Gamma^{\sigma,3}_{\alpha\beta} + \cdots, \;\; \Gamma^3_{\alpha\beta}(\varepsilon) = \varepsilon\partial_{\alpha\beta}\theta + \varepsilon^3\Gamma^{3,3}_{\alpha\beta} + \cdots \\
\Gamma^\sigma_{\alpha 3}(\varepsilon) &= -\varepsilon\partial_{\alpha\sigma}\theta + \varepsilon^3\Gamma^{\sigma,3}_{\alpha 3} + \cdots, \;\; \Gamma^3_{\alpha 3}(\varepsilon) = \Gamma^p_{33}(\varepsilon) = 0
\end{aligned}\right\} \tag{3.2.2}$$

$$\left.\begin{aligned}
A^{ijkl}(\varepsilon) &= A^{ijkl}(0) + \varepsilon^2 A^{ijkl,1}(x_1, x_2, x_3) + \varepsilon^4 A^{ijkl,2}(x_1, x_2, x_3) + \ldots \\
A^{ijkl}(\varepsilon)\sqrt{g(\varepsilon)} &= A^{ijkl}(0) + \varepsilon^2 B^{ijkl,1}(x_1, x_2, x_3) + \varepsilon^4 B^{ijkl,2}(x_1, x_2, x_3) + \ldots \\
A^{ijkl}(0) &= \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \\
A^{\alpha\beta\sigma\tau}(0) &= \lambda\delta^{\alpha\beta}\delta^{\sigma\tau} + \mu(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma}), \;\; A^{3333}(0) = (\lambda + 2\mu) \\
A^{\alpha\beta 33}(0) &= \lambda\delta^{\alpha\beta}, \;\; A^{33\sigma\tau}(0) = \lambda\delta^{\sigma\tau}, \;\; A^{\alpha 3\sigma 3}(0) = \mu\delta^{\alpha\sigma}
\end{aligned}\right\} \tag{3.2.3}$$

To justify the assumptions (3.1.3) and (3.1.5) we will follow the basic Ansatz of the asymptotic expansion method (cf. Lions (1973)). Write a priori $u(\varepsilon)$ as a formal expansion

$$u(\varepsilon) = \sum_{i\in\mathbb{Z}} \varepsilon^i u^i, \tag{3.2.4}$$

where $u^i \in H^1(\Omega)$. Equating to zero the factors of the successive powers $\varepsilon^p,\ p \geq 0$, we identify the successive terms u^i in the equation (3.2.1). For doing so, it is necessary to express the right-hand side of this equation in terms of power of ε. We are not assuming the existence of asymptotic expansions of the forces, but rather we are looking for the right asymptotic order of magnitude in ε. That is, we are trying to decide which assumption of the form (3.1.5) are appropriate.

Since the problem (3.2.1) is linear with respect to $u(\varepsilon)$, there is no restriction in assuming that expansion (3.2.4) begins with a term of order 0:

$$u(\varepsilon) = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \ldots \tag{3.2.5}$$

The boundary condition of place $u(\varepsilon) = 0$ on Γ_0 is imposed only on the first non-vanishing components of the expansion (3.2.5).

Then we have

$$e_{\sigma||\tau}(\varepsilon; u(\varepsilon)) = e^0_{\sigma||\tau}(u) + \varepsilon e^1_{\sigma||\tau}(u) + \varepsilon^2 e^2_{\sigma||\tau}(u) + \varepsilon^3 e^3_{\sigma||\tau}(u) + \cdots \tag{3.2.6}$$

where

$$\left.\begin{aligned} e^0_{\sigma||\tau}(u) &= \frac{1}{2}(\partial_\sigma u^0_\tau + \partial_\tau u^0_\sigma), \quad e^1_{\sigma||\tau}(u) = \frac{1}{2}(\partial_\sigma u^1_\tau + \partial_\tau u^1_\sigma) - u^0_3 \partial_{\sigma\tau}\theta \\ e^2_{\sigma||\tau}(u) &= \frac{1}{2}(\partial_\sigma u^2_\tau + \partial_\tau u^2_\sigma) - \Gamma^{\gamma,2}_{\sigma\tau} u^0_\gamma(\varepsilon) - \partial_{\sigma\tau}\theta u^1_3 \\ e^3_{\sigma||\tau}(u) &= \frac{1}{2}(\partial_\sigma u^3_\tau + \partial_\tau u^3_\sigma) - \Gamma^{\gamma,2}_{\sigma\tau} u^1_\gamma - \Gamma^{\gamma,3}_{\sigma\tau} u^0_\gamma - \partial_{\sigma\tau}\theta u^2_3 - \Gamma^{3,3}_{\sigma\tau} u^0_3 \end{aligned}\right\}, \tag{3.2.7}$$

$$e_{\sigma||3}(\varepsilon; u(\varepsilon)) = \varepsilon^{-1} e^{-1}_{\sigma||3}(u) + e^0_{\sigma||3}(u) + \varepsilon e^1_{\sigma||3}(u) + \varepsilon^2 e^2_{\sigma||3}(u) + \varepsilon^3 e^3_{\sigma||3}(u) + \cdots \tag{3.2.8}$$

where

$$\left.\begin{aligned} e^{-1}_{\sigma||3}(u) &= \frac{1}{2}\partial_3 u^0_\sigma, \;\; e^0_{\sigma||3}(u) = \frac{1}{2}(\partial_\sigma u^0_3 + \partial_3 u^1_\sigma), \;\; e^1_{\sigma||3}(u) = \frac{1}{2}(\partial_\sigma u^1_3 + \partial_3 u^2_\sigma) + u^0_\gamma \partial_{\sigma\gamma}\theta, \\ e^2_{\sigma||3}(u) &= \frac{1}{2}(\partial_\sigma u^2_3 + \partial_3 u^3_\sigma) + u^1_\gamma \partial_{\sigma\gamma}\theta, \;\; e^3_{\sigma||3}(u) = \frac{1}{2}(\partial_\sigma u^3_3 + \partial_3 u^4_\sigma) + u^2_\gamma \partial_{\sigma\gamma}\theta - \Gamma^{\gamma,3}_{\sigma 3} u^0_\gamma \end{aligned}\right\}, \tag{3.2.9}$$

$$e_{3||3}(\varepsilon; u(\varepsilon)) = \varepsilon^{-1} e^{-1}_{3||3}(u) + e^0_{3||3}(u) + \varepsilon e^1_{3||3}(u) + \varepsilon^2 e^2_{3||3}(u) + \varepsilon^3 e^3_{3||3}(u) + \ldots \tag{3.2.10}$$

where

$$\left.\begin{aligned} e^{-1}_{3||3}(u) = \partial_3 u^0_3, \;\; e^0_{3||3}(u) = \partial_3 u^1_3, \quad e^1_{3||3}(u) = \partial_3 u^2_3, \\ e^2_{3||3}(u) = \partial_3 u^3_3, \quad e^3_{3||3}(u) = \partial_3 u^4_3 \end{aligned}\right\}. \tag{3.2.11}$$

For any $v \in V$, we have

$$e_{\alpha||\beta}(\varepsilon; v) = e^0_{\alpha||\beta}(v) + \varepsilon e^1_{\alpha||\beta}(v) + \varepsilon^2 e^2_{\alpha||\beta}(v) + \varepsilon^3 e^3_{\alpha||\beta}(v) + \cdots \tag{3.2.12}$$

where

$$\left.\begin{aligned} e^0_{\alpha||\beta}(v) &= \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha), \;\; e^1_{\alpha||\beta}(v) = -v_3\partial_{\alpha\beta}\theta \\ e^2_{\alpha||\beta}(v) &= -\Gamma^{\gamma,2}_{\alpha\beta}v_\gamma, \;\; e^3_{\alpha||\beta}(v) = -\Gamma^{\gamma,3}_{\alpha\beta}v_\gamma - \Gamma^{3,3}_{\alpha\beta}v_3 \end{aligned}\right\}, \tag{3.2.13}$$

$$e_{\alpha||3}(\varepsilon; v) = \varepsilon^{-1}e^{-1}_{\alpha||3}(v) + e^0_{\alpha||3}(v) + \varepsilon e^1_{\alpha||3}(v) + \varepsilon^2 e^2_{\alpha||3}(v) + \varepsilon^3 e^3_{\alpha||3}(v) + \cdots \tag{3.2.14}$$

where

$$\left.\begin{aligned} e^{-1}_{\alpha||3}(v) = \frac{1}{2}\partial_3 v_\alpha, \;\; e^0_{\alpha||3}(v) = \frac{1}{2}\partial_\alpha v_3, \;\; e^1_{\alpha||3}(v) = v_\gamma\partial_{\alpha\gamma}\theta, \\ e^2_{\alpha||3}(v) = 0, \;\; e^3_{\alpha||3}(v) = -\Gamma^{\gamma,3}_{\alpha 3}v_\gamma \end{aligned}\right\}, \tag{3.2.15}$$

$$e_{3||3}(\varepsilon; v) = \frac{1}{\varepsilon}\partial_3 v_3. \tag{3.2.16}$$

Equation (3.2.1) can be written as

$$\begin{aligned} &\int_\Omega \Big\{ A^{\alpha\beta\sigma\tau}(\varepsilon)e_{\sigma||\tau}(\varepsilon; u(\varepsilon))e_{\alpha||\beta}(\varepsilon; v)\sqrt{g(\varepsilon)} + 4A^{\alpha3\sigma3}(\varepsilon)e_{\sigma||3}(\varepsilon; u(\varepsilon))e_{\alpha||3}(\varepsilon; v)\sqrt{g(\varepsilon)} \\ &\quad + A^{\alpha\beta33}(\varepsilon)e_{3||3}(\varepsilon; u(\varepsilon))e_{\alpha||\beta}(\varepsilon; v)\sqrt{g(\varepsilon)} + A^{33\sigma\tau}(\varepsilon)e_{\sigma||\tau}(\varepsilon; u(\varepsilon))e_{3||3}(\varepsilon; v)\sqrt{g(\varepsilon)} \\ &\quad + A^{3333}(\varepsilon)e_{3||3}(\varepsilon; u(\varepsilon))e_{3||3}(\varepsilon; v)\sqrt{g(\varepsilon)}\Big\}\, dx \\ &\quad = \int_\Omega f_i(\varepsilon)v_i\sqrt{g(\varepsilon)}dx. \end{aligned} \tag{3.2.17}$$

Using (3.2.5) - (3.2.16), the above equation becomes

$$\begin{aligned} &\int_\Omega \left\{ \frac{1}{\varepsilon^2}B_{-2}(u,v) + \frac{1}{\varepsilon}B_{-1}(u,v) + B_0(u,v) + \varepsilon B_1(u,v) + \varepsilon^2 B_2(u,v)\right\} dx + O(\varepsilon^3) \\ &= \int_\Omega f_i(\varepsilon)v_i dx + \varepsilon^2\int_\Omega f_i(\varepsilon)v_i g^1(x_1,x_2,x_3)dx + O(\varepsilon^4) \end{aligned} \tag{3.2.18}$$

where

$$B_{-2}(u,v) = \left\{4A^{\alpha3\sigma3}(0)e^{-1}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v) + A^{3333}(0)e^{-1}_{3||3}(u)\partial_3 v_3\right\}, \tag{3.2.19}$$

$$B_{-1}(u,v) = \Big\{ 4A^{\alpha 3\sigma 3}(0)(e^{-1}_{\sigma||3}(u)e^{0}_{\alpha||3}(v) + e^{0}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)) + A^{33\sigma\tau}(0)e^{0}_{\sigma||\tau}(u)\partial_3 v_3$$
$$+ A^{3333}(0)e^{0}_{3||3}(u)\partial_3 v_3 + A^{\alpha\beta 33}(0)e^{-1}_{3||3}(u)e^{0}_{\alpha||\beta}(v) \Big\}, \qquad (3.2.20)$$

$$B_0(u,v) = \Big\{ A^{\alpha\beta\sigma\tau}(0)e^{0}_{\sigma||\tau}(u)e^{0}_{\alpha||\beta}(v) + 4A^{\alpha 3\sigma 3}(0)(e^{-1}_{\sigma||3}(u)e^{1}_{\alpha||3}(v) + e^{0}_{\sigma||3}(u)e^{0}_{\alpha||3}(v)$$
$$+ e^{1}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)) + 4B^{\alpha 3\sigma 3,1}e^{-1}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v) + A^{33\sigma\tau}(0)e^{1}_{\sigma||\tau}(u)\partial_3 v_3$$
$$+ A^{3333}(0)e^{1}_{3||3}(u)\partial_3 v_3 + B^{3333,1}e^{-1}_{3||3}(u)\partial_3 v_3$$
$$+ A^{\alpha\beta 33}(0)(e^{-1}_{3||3}(u)e^{1}_{\alpha||\beta}(v) + e^{0}_{3||3}(u)e^{0}_{\alpha||\beta}(v)) \Big\}, \qquad (3.2.21)$$

$$B_1(u,v) = \Big\{ A^{\alpha\beta\sigma\tau}(0)(e^{0}_{\sigma||\tau}(u)e^{1}_{\alpha||\beta}(v) + e^{1}_{\sigma||\tau}(u)e^{0}_{\alpha||\beta}(v)) + B^{\alpha\beta 33,1}e^{-1}_{3||3}(u)e^{0}_{\alpha||\beta}(v)$$
$$+ 4A^{\alpha 3\sigma 3}(0)(e^{-1}_{\sigma||3}(u)e^{2}_{\alpha||3}(v) + e^{1}_{\sigma||3}(u)e^{0}_{\alpha||3}(v) + e^{0}_{\sigma||3}(u)e^{1}_{\alpha||3}(v)$$
$$+ e^{2}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)) + 4B^{\alpha 3\sigma 3,1}(e^{-1}_{\sigma||3}(u)e^{0}_{\alpha||3}(v) + e^{0}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v))$$
$$+ A^{33\sigma\tau}(0)e^{2}_{\sigma||\tau}(u)\partial_3 v_3 + B^{33\sigma\tau,1}e^{0}_{\sigma||\tau}(u)\partial_3 v_3 + A^{3333}(0)e^{2}_{3||3}(u)\partial_3 v_3$$
$$+ B^{3333,1}e^{0}_{3||3}(u)\partial_3 v_3 + A^{\alpha\beta 33}(0)(e^{1}_{3||3}(u)e^{0}_{\alpha||\beta}(v) + e^{0}_{3||3}(u)e^{1}_{\alpha||\beta}(v))$$
$$+ e^{-1}_{3||3}(u)e^{2}_{\alpha||\beta}(v)) \Big\}, \qquad (3.2.22)$$

$$B_2(u,v) = \big\{ A^{\alpha\beta\sigma\tau}(0)(e^{2}_{\sigma||\tau}(u)e^{0}_{\alpha||\beta}(v) + e^{1}_{\sigma||\tau}(u)e^{1}_{\alpha||\beta}(v) + e^{0}_{\sigma||\tau}(u)e^{2}_{\alpha||\beta}(v))$$
$$+ B^{\alpha\beta\sigma\tau,1}(0)e^{0}_{\sigma||\tau}(0)e^{0}_{\alpha||\beta}(v) + 4B^{\alpha 3\sigma 3,1}(e^{-1}_{\sigma||3}(u)e^{1}_{\alpha||3}(v) + e^{0}_{\sigma||3}(u)e^{0}_{\alpha||3}(v)$$
$$+ e^{1}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)) + 4A^{\alpha 3\sigma 3}(0)(e^{1}_{\sigma||3}(u)e^{1}_{\alpha||3}(v) + e^{2}_{\sigma||3}(u)e^{0}_{\alpha||3}(v)$$
$$+ e^{-1}_{\sigma||3}(u)e^{3}_{\alpha||3}(v) + e^{0}_{\sigma||3}(u)e^{2}_{\alpha||3}(v) + e^{3}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)) + A^{33\sigma\tau}e^{3}_{\sigma||\tau}(u)\partial_3 v_3$$
$$+ B^{33\sigma\tau,1}e^{1}_{\sigma||\tau}(u)\partial_3 v_3 + A^{3333}(0)e^{3}_{3||3}(u)\partial_3 v_3 + 4B^{\alpha 3\sigma 3,2}e^{-1}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)$$
$$+ B^{\alpha\beta 33,1}(e^{0}_{3||3}(u)e^{0}_{\alpha||\beta}(v) + e^{-1}_{3||3}(u)e^{1}_{\alpha||\beta}(v) + A^{\alpha\beta 33}(0)(e^{1}_{3||3}(u)e^{1}_{\alpha||\beta}(v)$$
$$+ e^{2}_{3||3}(u)e^{0}_{\alpha||\beta}(v) + e^{0}_{3||3}(u)e^{2}_{\alpha||\beta}(v) + e^{-1}_{3||3}(u)e^{3}_{\alpha||\beta}(v))$$
$$+ B^{3333,1}e^{1}_{3||3}(u)\partial_3 v_3 + B^{3333,2}e^{-1}_{3||3}(u)\partial_3 v_3 \Big\}. \qquad (3.2.23)$$

By close look at the different powers of ε that appear in the variational problem (3.2.18) shows that there is no need to consider forces whose order would be inferior to $f(\varepsilon) = \dfrac{1}{\varepsilon^2} f^{-2}$.

3.3 Identification of the Successive Terms of the Asymptotic Expansion $u(\varepsilon)$:

We define the space

$$W = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \ \Gamma_0\}.$$

Lemma 3.3.1. *For the smooth function $f : \bar{\Omega} \to \mathbb{R}$ the problem: Find $u \in W$ such that*

$$\int_\Omega \partial_3 u \partial_3 v dx = \int_\Omega f v dx \tag{3.3.1}$$

for all $v \in W$, has solutions only if the following compatibility conditions are satisfied

$$\int_{-1}^{1} f(x_1, x_2, t) dt = 0 \textit{ a.e in } \omega, \tag{3.3.2}$$

$$f = 0 \textit{ on } \Gamma_0. \tag{3.3.3}$$

If the above relations holds, then the solutions are of the form

$$u(x_1, x_2, x_3) = \zeta(x_1, x_2) - \int_{-1}^{x_3} \left(\int_{-1}^{s} f(x_1, x_2, t)\, dt \right) ds \tag{3.3.4}$$

where ζ is an arbitrary function in $V_3(\omega)$.

Thus, for general f problem (3.3.1) *has no solution.*

Proof. The variational equation (3.3.1) imply that $\partial_{33} u = -f$ in the distribution sense; this implies u takes the following expression (cf. Le Dret (1991)).

$$u(x) = \zeta(x_1, x_2) + x_3 \eta(x_1, x_2) - \int_{-1}^{x_3} \left(\int_{-1}^{s} f(x_1, x_2, t) dt \right) ds \ \text{ in } \Omega.$$

Therefore, for all $v \in W$,

$$\begin{aligned} \int_\Omega \partial_3 u \partial_3 v dx &= \int_\Omega \eta \partial_3 v dx - \int_\Omega \left(\int_{-1}^{x_3} f(x_1, x_2, t)\, dt \right) \partial_3 v dx \\ &= \int_{\Gamma_+} \eta v da - \int_{\Gamma_-} \eta v da + \int_\Omega f v dx - \int_{\Gamma_+} \left(\int_{-1}^{1} f(x_1, x_2, t)\, dt \right) v da, \end{aligned}$$

and equation (3.3.1) is satisfied only if, for all $w \in L^2(\omega)$,

$$\int_{\Gamma_+} \left(\eta - \int_{-1}^{1} f(x_1, x_2, t)\, dt\right) w da = 0 \quad \text{and} \quad - \int_{\Gamma_-} \eta w da = 0,$$

which yields condition (3.3.2). The boundary condition $u = 0$ on Γ_0 implies

$$\zeta(x_1, x_2) + x_3\eta(x_1, x_2) - \int_{-1}^{x_3} \left(\int_{-1}^{s} f(x_1, x_2, t)dt\right) ds = 0 \text{ on } \Gamma_0.$$

Hence the functions

$$h : x \in \Omega \to h(x_1, x_2, x_3) = x_3\eta(x_1, x_2) - \int_{-1}^{x_3} \left(\int_{-1}^{s} f(x_1, x_2, t)dt\right) ds$$

must be independent of x_3 for all $(x_1, x_2) \in \gamma_0$. Hence

$$\partial_3 h(x_1, x_2, x_3) = \eta(x_1, x_2) - \int_{-1}^{x_3} f(x_1, x_2, t)dt = 0 \text{ on } \Gamma_0.$$

and

$$\partial_{33} h(x_1, x_2, x_3) = f(x_1, x_2, x_3) = 0 \text{ on } \Gamma_0. \qquad \square$$

Lemma 3.3.1 will be used in the remainder of this work in the following way; Suppose that equation (3.3.1) (or at least its right-hand side) is the one obtained by equating to 0 the coefficient of $\varepsilon^k,\ k \geq -2$, in the equation (3.2.18) and that f represent the same component, indexed by i, of a general system of forces $f(\varepsilon)$ acting on Ω, i.e.,

$$f_i(\varepsilon) = \varepsilon^k f_i^k, \quad f_i^k = f \text{ on } \Omega,$$

the compatibility condition (3.3.2)-(3.3.3) mean that f cannot be chosen arbitrarily. Therefore we are led to "try" new assumptions of a higher order on the data: $f_i(\varepsilon) = \varepsilon^{k+1} f_i^{k+1}$. From these, we infer that $f = 0$ in Ω, and by Lemma 3.3.1, that $\partial_3 u = 0$ in Ω.

Our main result is the following.

Theorem 3.3.2. *If the first four terms $u^i,\ 0 \leq i \leq 3$ of the expansion* (3.2.5) *exists, we* *must assume that the applied forces are of the form*

$$f_\alpha^\varepsilon(x^\varepsilon) = f_\alpha(\varepsilon)(x) = \varepsilon f_\alpha^1(x) \text{ and } f_3^\varepsilon(x^\varepsilon) = f_3(\varepsilon)(x) = \varepsilon^2 f_3^2(x), \;\; f_\alpha^1, f_3^2 \in L^2(\Omega).$$

This implies that there exists a two dimensional vector field $\boldsymbol{\zeta} = (\zeta_\alpha^1, \zeta_3^0) : \bar{\omega} \to \mathbb{R}^3$ *satisfies the following variational problem*

$$-\int_\omega m^{\alpha\beta}(\boldsymbol{\zeta})\partial_{\alpha\beta}\eta_3 d\omega - \int_\omega \widetilde{n}^{\alpha\beta}(\boldsymbol{\zeta})\eta_3\partial_{\alpha\beta}\theta d\omega + \int_\omega \widetilde{n}^{\alpha\beta}(\boldsymbol{\zeta})\partial_\beta\eta_\alpha d\omega = \int_\omega p^i\eta_i d\omega - \int_\omega q^i\partial_\alpha\eta_3 d\omega \tag{3.3.5}$$

for all $\eta \in V(\omega)$,

where

$$m^{\alpha\beta}(\boldsymbol{\zeta}) = -\left\{\frac{4\lambda\mu}{3(\lambda+2\mu)}\Delta\zeta_3\delta_{\alpha\beta} + \frac{4\mu}{3}\partial_{\alpha\beta}\zeta_3\right\}, \tag{3.3.6}$$

$$\widetilde{n}^{\alpha\beta}(\boldsymbol{\zeta}) = \frac{4\lambda\mu}{(\lambda+2\mu)}\widetilde{e}_{\sigma\sigma}(\boldsymbol{\zeta})\delta_{\alpha\beta} + 4\mu\widetilde{e}_{\alpha\beta}(\boldsymbol{\zeta}), \tag{3.3.7}$$

$$\widetilde{e}_{\alpha\beta}(\boldsymbol{\zeta}) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) - \zeta_3\partial_{\alpha\beta}\theta, \tag{3.3.8}$$

$$p^\alpha = \int_{-1}^1 f_\alpha^1 dx_3, \qquad p^3 = \int_{-1}^1 f_3^2 dx_3, \qquad q^\alpha = \int_{-1}^1 x_3 f_\alpha^1 dx_3. \tag{3.3.9}$$

Proof. For clarity, the proof is broken into five steps

Step 1: If u^0 exists, then f^{-2} vanishes on Ω and there exists $\zeta^0(x_1, x_2) \in (H^1(\omega))^3$ independent of x_3 with $\zeta^0 = 0$ on γ_0, such that $u^0(x_1, x_2, x_3) = \zeta^0(x_1, x_2)$ in Ω.

Equating the coefficient of ε^{-2} on both sides of (3.2.18) leads to the problem : Find $u^0 \in V$ such that

$$\int_\Omega \left[4A^{\alpha3\sigma3}(0)e^{-1}_{\sigma||3}(u)e^{-1}_{\alpha||3}(v) + A^{3333}(0)e^{-1}_{3||3}(u)\partial_3 v_3\right] dx = \int_\Omega f_i^{-2}v_i dx \tag{3.3.10}$$

for all $v \in V$.

This implies

$$\int_\Omega \left[4\mu\delta^{\alpha\sigma}\,\frac{1}{2}\,\partial_3 u_\sigma^0\,\frac{1}{2}\partial_3 v_\alpha + (\lambda+2\mu)\,\partial_3 u_3^0\,\partial_3 v_3\right] dx = \int_\Omega f_i^{-2}v_i dx \text{ for all } v \in V. \tag{3.3.11}$$

This is a set of three variational problems of the form (3.3.1). Thus for general term

f^{-2}, there is no solution u^0. Therefore we must impose $f^{-2} = 0$ on Ω.

This implies $\partial_3 u^0_\sigma = 0$ and $\partial_3 u^0_3 = 0$. Hence

$$e^{-1}_{\sigma||3}(u) = \partial_3 u^0_\sigma = 0, \;\; e^{-1}_{3||3}(u) = \partial_3 u^0_3 = 0. \tag{3.3.12}$$

Also, $\partial_3 u^0 = 0$ implies that there exists a two dimensional vector field $\zeta^0(x_1, x_2) \in (H^1(\omega))^3$ independent of x_3 with $\zeta^0 = 0$ on γ_0, such that $u^0(x_1, x_2, x_3) = \zeta^0(x_1, x_2)$ in Ω.

Hence we have to try new orders for the forces , i.e., $f(\varepsilon) = \dfrac{1}{\varepsilon} f^{-1} \in L^2(\Omega)$.

Step 2: If $u^1 \in H^1(\Omega)$ exists, then $f^{-1} = 0$ on Ω and there exists a two dimensional vector field $\zeta^1 = (\zeta^1_i) \in H^1(\omega)$ such that

$$u^1_\sigma = \zeta^1_\sigma(x_1, x_2) - x_3 \;\; \partial_\sigma \zeta^0_3(x_1, x_2), \tag{3.3.13}$$

$$u^1_3 = \zeta^1_3(x_1, x_2) - x_3 \;\; \frac{\lambda}{2(\lambda + 2\mu)} \delta^{\sigma\tau} (\partial_\sigma \zeta^0_\tau + \partial_\tau \zeta^0_\sigma). \tag{3.3.14}$$

In addition, the two dimensional vector field $\zeta^0 = (\zeta^0_i)$ must satisfy the regularity condition: $\partial_\alpha \zeta^0_\alpha \in H^1(\omega), \; \partial_\alpha \zeta^0_3 \in H^1(\omega)$.

Equating the coefficient of ε^{-1} on both sides of (3.2.18) leads to the problem : Find $u^1 \in H^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega \Big\{ & 4A^{\alpha 3\sigma 3}(0)(e^{-1}_{\sigma||3}(u) e^0_{\alpha||3}(v) + e^0_{\sigma||3}(u) e^{-1}_{\alpha||3}(v)) + A^{33\sigma\tau}(0) e^0_{\sigma||\tau}(u) \partial_3 v_3 \\ & + A^{3333}(0) e^0_{3||3}(u) \partial_3 v_3 + A^{\alpha\beta 33}(0) e^{-1}_{3||3}(u) e^0_{\alpha||\beta}(v) \Big\} \, dx = \int_\Omega f^{-1}_i v_i dx \end{aligned} \tag{3.3.15}$$

for all $v \in V$. Using (3.3.12) we have

$$\begin{aligned} & \int_\Omega \Big\{ 4\mu \delta^{\alpha\sigma} \frac{1}{2} (\partial_\sigma u^0_3 + \partial_3 u^1_\sigma) \frac{1}{2} \partial_3 v_\alpha + \lambda \delta^{\sigma\tau} \frac{1}{2} (\partial_\sigma u^0_\tau + \partial_\tau u^0_\sigma) \partial_3 v_3 + (\lambda + 2\mu) \partial_3 u^1_3 \partial_3 v_3 \Big\} \, dx \\ & \quad = \int_\Omega f^{-1}_i v_i dx \;\; \text{for all} \;\; v \in V. \end{aligned} \tag{3.3.16}$$

This problem can be split into two decoupled problems of the form (3.3.1) viz.,

$$\int_\Omega \left\{ \lambda\delta^{\sigma\tau}\frac{1}{2}(\partial_\sigma u_\tau^0 + \partial_\tau u_\sigma^0) + (\lambda+2\mu)\partial_3 u_3^1 \right\} \partial_3 v_3 dx = \int_\Omega f_3^{-1} v_3 dx \tag{3.3.17}$$

for all $v_3 \in W$ and

$$\int_\Omega \mu\delta^{\alpha\sigma}\left(\partial_\sigma u_3^0 + \partial_3 u_\sigma^1\right)\partial_3 v_\alpha dx = \int_\Omega f_\alpha^{-1} v_\alpha dx \text{ for all } (v_\alpha) \in W \times W. \tag{3.3.18}$$

The above equations can be written as

$$\int_\Omega \partial_3 \left\{ \frac{x_3}{2}\lambda\delta^{\sigma\tau}(\partial_\sigma\zeta_\tau^0 + \partial_\tau\zeta_\sigma^0) + (\lambda+2\mu)u_3^1 \right\} \partial_3 v_3 dx = \int_\Omega f_3^{-1} v_3 dx \text{ for all } v_3 \in W \tag{3.3.19}$$

and

$$\int_\Omega \mu\delta^{\alpha\sigma}\partial_3\{x_3\partial_\sigma\zeta_3^0 + u_\sigma^1\}\partial_3 v_\alpha dx = \int_\Omega f_\alpha^{-1} v_\alpha dx \text{ for all } (v_\alpha) \in W \times W. \tag{3.3.20}$$

By applying Lemma 3.3.1, we notice that for general terms f_i^{-1} there is no solution u^1. Therefore we must impose $f_i^{-1} = 0$ which leads to

$$\partial_3\left(\frac{x_3}{2}\lambda\delta^{\sigma\tau}(\partial_\sigma\zeta_\tau^0 + \partial_\tau\zeta_\sigma^0) + (\lambda+2\mu)u_3^1\right) = 0 \text{ and } \partial_3(x_3\partial_\sigma\zeta_3^0 + u_\sigma^1) = 0. \tag{3.3.21}$$

This implies there exists a two dimensional vector field $\zeta^1 = (\zeta_i^1) \in H^1(\omega)$ such that

$$u_\sigma^1(x_1,x_2,x_3) = \zeta_\sigma^1(x_1,x_2) - x_3 \ \partial_\sigma\zeta_3^0(x_1,x_2)$$

and

$$u_3^1(x_1,x_2,x_3) = \zeta_3^1(x_1,x_2) - x_3 \ \frac{\lambda}{2(\lambda+2\mu)}\delta^{\sigma\tau}(\partial_\sigma\zeta_\tau^0 + \partial_\tau\zeta_\sigma^0).$$

Hence

$$\partial_3 u_3^1 = -\frac{\lambda}{2(\lambda+2\mu)}\delta^{\sigma\tau}(\partial_\sigma\zeta_\tau^0 + \partial_\tau\zeta_\sigma^0) \tag{3.3.22}$$

and

$$e_{\sigma||3}^0(u) = \partial_3 u_\sigma^1 + \partial_\sigma u_3^0 = 0. \tag{3.3.23}$$

Since $u^1 \in H^1(\Omega)$, it imposes the regularity conditions on $\partial_\alpha \zeta_\alpha^0$ and $\partial_\alpha \zeta_3^0$.

We now try new orders for the forces; i.e., $f(\varepsilon) = f^0 \in L^2(\Omega)$.

Step 3 : If $u^2 \in H^1(\Omega)$ exists, then $f_3^0 = 0$ and there exists $\zeta_3^2 \in H^1(\omega)$ such that

$$u_3^2 = \zeta_3^2(x_1, x_2) - \frac{\lambda}{\lambda + 2\mu} x_3[\partial_\sigma \zeta_\sigma^1(x_1, x_2) - \zeta_3^0 \partial_{\sigma\sigma}\theta] + \frac{\lambda}{\lambda + 2\mu}\frac{x_3^2}{2}\partial_{\sigma\sigma}\zeta_3^0(x_1, x_2). \tag{3.3.24}$$

In addition to that the vertical component ζ_3^0 and the horizontal component ζ_α^1 must satisfy the regularity condition $\partial_{\alpha\alpha}\zeta_3^0 \in H^1(\omega),\ \partial_\alpha \zeta_\alpha^0 \in H^1(\omega)$ respectively.

Equating the coefficient of ε^0 on both sides of (3.2.18) leads to the problem : Find $u^2 \in H^1(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \{A^{\alpha\beta\sigma\tau}(0) e^0_{\sigma||\tau}(u) e^0_{\alpha||\beta}(v) + 4A^{\alpha 3\sigma 3}(0)(e^{-1}_{\sigma||3}(u) e^1_{\alpha||3}(v) + e^0_{\sigma||3}(u) e^0_{\alpha||3}(v) \\
&+ e^1_{\sigma||3}(u) e^{-1}_{\alpha||3}(v)) + 4B^{\alpha 3\sigma 3,1} e^{-1}_{\sigma||3}(u) e^{-1}_{\alpha||3}(v) + A^{33\sigma\tau}(0) e^1_{\sigma||\tau}(u)\partial_3 v_3 \\
&+ A^{3333}(0) e^1_{3||3}(u)\partial_3 v_3 + B^{3333,1} e^{-1}_{3||3}(u)\partial_3 v_3 + A^{\alpha\beta 33}(0)(e^{-1}_{3||3}(u) e^1_{\alpha||\beta}(v) \\
&+ e^0_{3||3}(u) e^0_{\alpha||\beta}(v))\} dx = \int_\Omega f_i^0 v_i dx \ \text{ for all } \ v \in V.
\end{aligned} \tag{3.3.25}$$

Using (3.3.12) and (3.3.23), the above equation becomes

$$\begin{aligned}
&\int_\Omega \Bigg\{ A^{\alpha\beta\sigma\tau}(0)\frac{1}{2}(\partial_\sigma u_\tau^0 + \partial_\tau u_\sigma^0)\frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) + 4A^{\alpha 3\sigma 3}(0) e^1_{\sigma||3}(u)\frac{1}{2}\partial_3 v_\alpha \\
&\quad + \lambda\delta^{\sigma\tau}\left[\frac{1}{2}(\partial_\sigma u_\tau^1 + \partial_\tau u_\sigma^1) - u_3^0 \partial_{\sigma\tau}\theta\right]\partial_3 v_3 + (\lambda + 2\mu)\partial_3 u_3^2 \partial_3 v_3 \\
&\quad + A^{\alpha\beta 33}(0)\partial_3 u_3^1\,\frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha)\Bigg\} dx = \int_\Omega f_i^0 v_i dx \ \text{ for all } \ v \in V.
\end{aligned} \tag{3.3.26}$$

Taking $v_\alpha = 0$, we get

$$\int_\Omega \left\{(\lambda + 2\mu)\partial_3 u_3^2 + \lambda\delta^{\sigma\tau}\left[\frac{1}{2}(\partial_\sigma u_\tau^1 + \partial_\tau u_\sigma^1) - u_3^0\partial_{\sigma\tau}\theta\right]\right\}\partial_3 v_3 dx = \int_\Omega f_3^0 v_3 dx \tag{3.3.27}$$

for all $v_3 \in V$.

Taking $v_3 = \eta_3$ independent of x_3, we get

$$0 = \int_\omega \left(\int_{-1}^{1} f_3^0(x_1, x_2, t)dt \right) \eta_3 d\omega \ \text{ for all } \eta_3 \in H^1(\omega).$$

This implies $f_3^0 = 0$ in Ω. Hence (3.3.27) becomes

$$\int_\Omega \left\{ (\lambda + 2\mu)\partial_3 u_3^2 + \lambda\delta^{\sigma\tau} \left[\frac{1}{2}(\partial_\sigma u_\tau^1 + \partial_\tau u_\sigma^1) - u_3^0 \partial_{\sigma\tau}\theta \right] \right\} \partial_3 v_3 dx = 0 \ \text{ for all } \ v_3 \in W, \tag{3.3.28}$$

which implies

$$(\lambda + 2\mu)\partial_3 u_3^2 + \lambda\delta^{\sigma\tau} \left(\frac{1}{2}(\partial_\sigma u_\tau^1 + \partial_\tau u_\sigma^1) - u_3^0 \partial_{\sigma\tau}\theta \right) = 0. \tag{3.3.29}$$

Hence

$$\begin{aligned} \partial_3 u_3^2 &= -\frac{\lambda}{\lambda + 2\mu}\delta^{\sigma\tau} \left(\frac{1}{2}(\partial_\sigma u_\tau^1 + \partial_\tau u_\sigma^1) - u_3^0 \partial_{\sigma\tau}\theta \right) \\ &= -\frac{\lambda}{\lambda + 2\mu} \left(\partial_\sigma u_\sigma^1 - u_3^0 \partial_{\sigma\sigma}\theta \right) \\ &= -\frac{\lambda}{\lambda + 2\mu} \left(\partial_\sigma \zeta_\sigma^1(x_1, x_2) - x_3 \partial_{\sigma\sigma}\zeta_3^0(x_1, x_2) - \zeta_3^0 \partial_{\sigma\sigma}\theta \right). \end{aligned} \tag{3.3.30}$$

Therefore there exists $\zeta_3^2 \in H^1(\omega)$ such that

$$u_3^2 = \zeta_3^2(x_1, x_2) - \frac{\lambda}{\lambda + 2\mu} x_3 [\partial_\sigma \zeta_\sigma^1(x_1, x_2) - \zeta_3^0 \partial_{\sigma\sigma}\theta] + \frac{\lambda}{\lambda + 2\mu} \frac{x_3^2}{2} \partial_{\sigma\sigma}\zeta_3^0(x_1, x_2).$$

This proves (3.3.24). Since $u_3^2 \in H^1(\Omega)$, it imposes the regularity conditions on $\partial_{\alpha\alpha}\zeta_3^0$ and $\partial_\alpha \zeta_\alpha^1$.

Choosing test functions of the form $(\eta_1, \eta_2, 0) \in V(\omega)$ in equation (3.3.26), we get

$$\begin{aligned} &\int_\Omega \left\{ A^{\alpha\beta\sigma\tau}(0)\frac{1}{2}(\partial_\sigma u_\tau^0 + \partial_\tau u_\sigma^0)\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) + A^{\alpha\beta 33}(0)\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha)\partial_3 u_3^1 \right\} dx \\ &\quad = \int_\omega \left(\int_{-1}^{1} f_\alpha^0(x_1, x_2, t)dt \right) \eta_\alpha d\omega \ \ \forall \ \ \eta_\alpha \in V_H(\omega). \end{aligned} \tag{3.3.31}$$

Using (3.3.22), the above equation become

$$\int_\Omega \Big\{ A^{\alpha\beta\sigma\tau}(0)\frac{1}{4}(\partial_\sigma u_\tau^0 + \partial_\tau u_\sigma^0)(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha)$$
$$+A^{\alpha\beta 33}(0)\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha)\left(-\frac{\lambda}{\lambda+2\mu}\frac{1}{2}\delta^{\sigma\tau}(\partial_\sigma \zeta_\tau^0 + \partial_\tau \zeta_\sigma^0)\right)\Big\}\, dx$$
$$= \int_\omega \left(\int_{-1}^{1} f_\alpha^0(x_1,x_2,t)dt\right)\eta_\alpha d\omega \quad \text{for all } (\eta_\alpha, 0) \in V(\omega). \tag{3.3.32}$$

Substituting the values of $A^{\alpha\beta\sigma\tau}(0),\ A^{\alpha\beta 33}(0)$ given in (3.2.3), we get

$$\int_\Omega \left\{ \frac{2\mu\lambda}{\lambda+2\mu} e_{\sigma\sigma}(\zeta^0)e_{\alpha\alpha}(\eta) + 2\mu e_{\alpha\beta}(\zeta^0)e_{\alpha\beta}(\eta)\right\} dx = \int_\omega \left(\int_{-1}^{1} f_\alpha^0(x_1,x_2,t)dt\right)\eta_\alpha d\omega \tag{3.3.33}$$

for all $(\eta_\alpha, 0) \in V(\omega)$.

This problem has a unique solution $\zeta_\alpha^0 \in V_H(\omega)$ for $f_\alpha^0 \in L^2(\Omega)$.

For test functions $v = (-x_3\partial_1\eta_3, -x_3\partial_2\eta_3, 0) \in V$ with η_α, η_3 are independent of x_3, the variational problem (3.3.26) reduces to

$$\int_\Omega \Big\{ A^{\alpha\beta\sigma\tau}(0)e^0_{\sigma||\tau}(u)\frac{1}{2}\left[\partial_\alpha(-x_3\partial_\beta\eta_3) + \partial_\beta(-x_3\partial_\alpha\eta_3)\right] + 4A^{\alpha 3\sigma 3}(0)e^1_{\sigma||3}(u)\frac{1}{2}\partial_3(-x_3\partial_\alpha\eta_3)$$
$$+A^{\alpha\beta 33}(0)\partial_3 u_3^1\frac{1}{2}\left[\partial_\alpha(-x_3\partial_\beta\eta_3) + \partial_\beta(-x_3\partial_\alpha\eta_3)\right]\Big\}\, dx = -\int_\Omega x_3 f_\alpha^0\partial_\alpha\eta_3 dx \tag{3.3.34}$$

for all $\eta_3 \in V_3(\omega)$.

Since $e^0_{\sigma||\tau}(u)$ and $\partial_3 u_3^1$ are independent of x_3, the above equation reduces to

$$\int_\Omega 4A^{\alpha 3\sigma 3}(0)e^1_{\sigma||3}(u)\frac{1}{2}\partial_\alpha\eta_3 dx = \int_\Omega x_3 f_\alpha^0 \partial_\alpha \eta_3 dx \quad \forall\, \eta_3 \in V_3(\omega). \tag{3.3.35}$$

We now try new orders for the vertical component of the forces which are different from horizontal components as below

$$f_\alpha(\varepsilon) = f_\alpha^0 \in L^2(\Omega) \ \text{ and } \ f_3(\varepsilon) = \varepsilon f_3^1 \in L^2(\Omega). \tag{3.3.36}$$

Step 4 : If $u^3 \in H^1(\Omega)$ exists, then $f_3^1 = f_\alpha^0 = 0$ on Ω. In addition

(i) $\zeta_\alpha^0 = 0$,

(ii) (ζ_α^1) satisfies

$$\int_\Omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\sigma\sigma}(\zeta^1) e_{\alpha\alpha}(\eta) + 2\mu e_{\alpha\beta}(\zeta^1) e_{\alpha\beta}(\eta) \right] dx$$
$$= \int_\Omega \left(\frac{2\lambda\mu}{\lambda+2\mu} \zeta_3^0 \partial_{\sigma\sigma}\theta e_{\alpha\alpha}(\eta) + 2\mu\zeta_3^0 \partial_{\alpha\beta}\theta e_{\alpha\beta}(\eta) \right) dx + \int_\Omega f_\alpha^1 \eta_\alpha dx \quad (3.3.37)$$

for all $\eta_\alpha \in V_H(\omega)$.

Equating the coefficient of ε on both sides of (3.2.18) leads to the problem : Find $u^3 \in H^1(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \{ A^{\alpha\beta\sigma\tau}(0)(e^0_{\sigma||\tau}(u)e^1_{\alpha||\beta}(v) + e^1_{\sigma||\tau}(u)e^0_{\alpha||\beta}(v)) \\
&\quad + 4A^{\alpha3\sigma3}(0)(e^{-1}_{\sigma||3}(u)e^2_{\alpha||3}(v) + e^1_{\sigma||3}(u)e^0_{\alpha||3}(v) + e^0_{\sigma||3}(u)e^1_{\alpha||3}(v) + e^2_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)) \\
&\quad + 4B^{\alpha3\sigma3,1}(e^{-1}_{\sigma||3}(u)e^0_{\alpha||3}(v) + e^0_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)) + A^{33\sigma\tau}(0)e^2_{\sigma||\tau}(u)\partial_3 v_3 \\
&\quad + B^{33\sigma\tau,1}e^0_{\sigma||\tau}(u)\partial_3 v_3 + A^{3333}(0)e^2_{3||3}(u)\partial_3 v_3 + B^{3333,1}e^0_{3||3}(u)\partial_3 v_3 \\
&\quad + A^{\alpha\beta33}(0)(e^1_{3||3}(u)e^0_{\alpha||\beta}(v) + e^{-1}_{3||3}(u)e^2_{\alpha||\beta}(v) + e^0_{3||3}(u)e^1_{\alpha||\beta}(v))) \\
&\quad + B^{\alpha\beta33,1}e^{-1}_{3||3}(u)e^0_{\alpha||\beta}(v) \} dx \\
&\quad = \int_\Omega f_i^1 v_i dx \quad \forall\, v \in V. && (3.3.38)
\end{aligned}$$

Using (3.3.12) and (3.3.23), the above equation becomes

$$\begin{aligned}
&\int_\Omega \{ A^{\alpha\beta\sigma\tau}(0)(e^0_{\sigma||\tau}(u)e^1_{\alpha||\beta}(v) + e^1_{\sigma||\tau}(u)e^0_{\alpha||\beta}(v)) + 4A^{\alpha3\sigma3}(0)(e^1_{\sigma||3}(u)e^0_{\alpha||3}(v) \\
&+ e^2_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)) + A^{33\sigma\tau}(0)e^2_{\sigma||\tau}(u)\partial_3 v_3 + B^{33\sigma\tau,1}e^0_{\sigma||\tau}(u)\partial_3 v_3 + B^{3333,1}e^0_{3||3}(u)\partial_3 v_3 \\
&+ A^{3333}(0)e^2_{3||3}(u)\partial_3 v_3 + A^{\alpha\beta33}(0)(e^1_{3||3}(u)e^0_{\alpha||\beta}(v) + e^0_{3||3}(u)e^1_{\alpha||\beta}(v))) \} dx \\
&= \int_\Omega f_i^1 v_i dx \quad \forall\, v \in V. && (3.3.39)
\end{aligned}$$

If we consider the assumptions (3.3.36) on the forces, we realize that the problem (3.3.39) must be considered only for test functions with vanishing horizontal components (since the horizontal components of force f^1_α not yet introduced). Hence we have

$$\begin{aligned}
&\int_\Omega \Big\{ A^{\alpha\beta\sigma\tau}(0)e^0_{\sigma||\tau}(u)(-v_3\partial_{\alpha\beta}\theta) + 4A^{\alpha3\sigma3}(0)e^1_{\sigma||3}(u)\frac{1}{2}\partial_\alpha v_3 \\
&\quad + A^{33\sigma\tau}(0)e^2_{\sigma||\tau}(u)\partial_3 v_3 + A^{3333}(0)\partial_3 u^3_3\partial_3 v_3 + A^{\alpha\beta33}(0)e^0_{3||3}(u)(-v_3\partial_{\alpha\beta}\theta) \\
&\quad + B^{3333,1}e^0_{3||3}(u)\partial_3 v_3 + B^{33\sigma\tau,1}e^0_{\sigma||\tau}(u)\partial_3 v_3\Big\}\, dx = \int_\Omega f^1_3 v_3 dx \quad \forall\, v_3 \in W.
\end{aligned}$$

Taking $v_3 = \eta_3(x_1, x_2) \in V_3(\omega)$ we get

$$\begin{aligned}
&\int_\Omega \Big\{ A^{\alpha\beta\sigma\tau}(0)e^0_{\sigma||\tau}(u)(-\eta_3\partial_{\alpha\beta}\theta) + 4A^{\alpha3\sigma3}(0)e^1_{\sigma||3}(u)\frac{1}{2}\partial_\alpha \eta_3 \\
&\quad + A^{\alpha\beta33}(0)e^0_{3||3}(u)(-\eta_3\partial_{\alpha\beta}\theta)\Big\}\, dx = \int_\Omega f^1_3 \eta_3 dx \quad \forall\, \eta_3 \in V_3(\omega). \qquad (3.3.40)
\end{aligned}$$

From the equations (3.3.35) and (3.3.40), we have

$$\begin{aligned}
&-\int_\Omega \left(A^{\alpha\beta\sigma\tau}(0)e^0_{\sigma||\tau}(u)(\partial_{\alpha\beta}\theta) + A^{\alpha\beta33}(0)e^0_{3||3}(u)(\partial_{\alpha\beta}\theta)\right)\eta_3 dx \\
&\quad = \int_\Omega f^1_3(x_1, x_2, x_3)\eta_3 dx - \int_\Omega x_3 f^0_\alpha(x_1, x_2, x_3)\partial_\alpha \eta_3 dx \quad \forall\, \eta_3 \in V(\omega). \quad (3.3.41)
\end{aligned}$$

This implies

$$\left(A^{\alpha\beta\sigma\tau}(0)e^0_{\sigma||\tau}(u) + A^{\alpha\beta33}(0)e^0_{3||3}(u)\right)\partial_{\alpha\beta}\theta = f^1_3 + x_3\partial_\alpha f^0_\alpha.$$

The left hand side is function of x_1 and x_2 only, whereas the right hand side can be any arbitrary function.

This lead to impose $f^0_\alpha = f^1_3 = 0$ on Ω which implies $\zeta^0_\alpha = 0$ in ω by (3.3.33).

This in turn gives

$$e^0_{\sigma||\tau}(u) = 0 \ \text{ and } \ u^1_3 = \zeta^1_3(x_1, x_2) \ \text{ and hence } \ e^0_{3||3}(u) = \partial_3 u^1_3 = 0. \qquad (3.3.42)$$

Thus, we have to try with new order for the forces

$$f_\alpha(\varepsilon) = \varepsilon f^1_\alpha \in L^2(\Omega) \quad \text{and} \quad f_3(\varepsilon) = \varepsilon^2 f^2_3 \in L^2(\Omega). \qquad (3.3.43)$$

Using $f_\alpha^0 = 0$ in (3.3.35), we get

$$\int_\Omega 4A^{\alpha 3\sigma 3}(0) e^1_{\sigma||3}(u) \frac{1}{2}\partial_\alpha \eta_3 dx = 0 \quad \forall\, \eta_3 \in V_3(\omega). \tag{3.3.44}$$

Note that since $\zeta_\alpha^0 = 0$ in ω, the horizontal components of the leading term of the expansion becomes u_α^1. The boundary condition of place reads $u_\alpha^1 = 0$ on Γ_0, or equivalently $\partial_\alpha \zeta_3^0 = \zeta_\alpha^1 = 0$ on γ_0. This implies that $\zeta = (\zeta_\alpha^1, \zeta_3^0) \in V(\omega)$.

Taking $v = (\eta_1, \eta_2, 0) \in V(\omega)$ with η_α are independent of x_3 in (3.3.39), we get

$$\begin{aligned}
&\int_\Omega \left\{ A^{\alpha\beta\sigma\tau}(0) e^1_{\sigma||\tau}(u) \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) + A^{\alpha\beta 33}(0) e^1_{3||3}(u) \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) \right\} dx \\
&\qquad = \int_\Omega f_\alpha^1 \eta_\alpha dx.
\end{aligned} \tag{3.3.45}$$

Using (3.2.3), (3.2.7), (3.3.13) and (3.3.30), the left hand side of above equation becomes

$$\begin{aligned}
&\int_\Omega \left(\left(\lambda \delta^{\alpha\beta}\delta^{\sigma\tau} e^1_{\sigma||\tau}(u) + \lambda \delta^{\alpha\beta} e^1_{3||3}(u) \right) + \mu(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\tau}) e^1_{\sigma||\tau}(u) \right) e_{\alpha\beta}(\eta) dx \\
&\quad = \int_\Omega \left(\lambda \delta^{\alpha\beta} e^1_{p||p}(u) + 2\mu e^1_{\alpha||\beta}(u) \right) e_{\alpha\beta}(\eta) dx.
\end{aligned}$$

Now

$$\begin{aligned}
e^1_{p||p}(u) &= e^1_{1||1}(u) + e^1_{2||2}(u) + e^1_{3||3}(u) \\
&= (\partial_1 \zeta_1^1 - x_3 \partial_{11}\zeta_3^0 - \zeta_3^0 \partial_{11}\theta) + (\partial_2 \zeta_2^1 - x_3 \partial_{22}\zeta_3^0 - \zeta_3^0 \partial_{22}\theta) \\
&\quad - \frac{\lambda}{\lambda + 2\mu} \left(\partial_\sigma \zeta_\sigma^1 - x_3 \partial_{\sigma\sigma}\zeta_3^0 - \zeta_3^0 \partial_{\sigma\sigma}\theta \right) \\
&= (\partial_\sigma \zeta_\sigma^1 - x_3 \partial_{\sigma\sigma}\zeta_3^0 - \zeta_3^0 \partial_{\sigma\sigma}\theta) - \frac{\lambda}{\lambda + 2\mu} \left(\partial_\sigma \zeta_\sigma^1 - x_3 \partial_{\sigma\sigma}\zeta_3^0 - \zeta_3^0 \partial_{\sigma\sigma}\theta \right) \\
&= \frac{2\mu}{\lambda + 2\mu} \left(\partial_\sigma \zeta_\sigma^1 - x_3 \Delta \zeta_3^0 - \zeta_3^0 \partial_{\sigma\sigma}\theta \right).
\end{aligned} \tag{3.3.46}$$

Hence, left hand side of (3.3.45) becomes

$$\int_\Omega \Bigg\{ \left(\frac{2\lambda\mu}{\lambda+2\mu} \left(e_{\sigma\sigma}(\zeta^1) - x_3 \Delta\zeta_3^0 - \zeta_3^0 \partial_{\sigma\sigma}\theta \right) \right) \delta_{\alpha\beta}$$
$$+2\mu \left(\left[e_{\alpha\beta}(\zeta^1) - \zeta_3^0 \partial_{\alpha\beta}\theta \right] - x_3 \partial_{\alpha\beta}\zeta_3^0 \right) \Bigg\} e_{\alpha\beta}(\eta)$$
$$= \int_\Omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\sigma\sigma}(\zeta^1) e_{\alpha\alpha}(\eta) + 2\mu e_{\alpha\beta}(\zeta^1) e_{\alpha\beta}(\eta) \right] dx$$
$$- \left(\int_{-1}^{1} x_3 dx_3 \right) \int_\omega \frac{2\lambda\mu}{\lambda+2\mu} \Delta\zeta_3^0 e_{\alpha\alpha}(\eta) d\omega$$

$$- \int_\Omega \left(\frac{2\lambda\mu}{\lambda+2\mu} \zeta_3^0 \partial_{\sigma\sigma}\theta e_{\alpha\alpha}(\eta) + 2\mu\zeta_3^0 \partial_{\alpha\beta}\theta e_{\alpha\beta}(\eta) \right) dx$$
$$- \left(\int_{-1}^{1} x_3 dx_3 \right) \int_\omega 2\mu \partial_{\alpha\beta}\zeta_3^0 e_{\alpha\beta}(\eta) d\omega$$
$$= \int_\Omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\sigma\sigma}(\zeta^1) e_{\alpha\alpha}(\eta) + 2\mu e_{\alpha\beta}(\zeta^1) e_{\alpha\beta}(\eta) \right] dx$$
$$- \int_\Omega \left(\frac{2\lambda\mu}{\lambda+2\mu} \zeta_3^0 \partial_{\sigma\sigma}\theta e_{\alpha\alpha}(\eta) + 2\mu\zeta_3^0 \partial_{\alpha\beta}\theta e_{\alpha\beta}(\eta) \right) dx.$$

Therefore (3.3.45) becomes

$$\int_\Omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\sigma\sigma}(\zeta^1) e_{\alpha\alpha}(\eta) + 2\mu e_{\alpha\beta}(\zeta^1) e_{\alpha\beta}(\eta) \right] dx$$
$$= \int_\Omega \left(\frac{2\lambda\mu}{\lambda+2\mu} \zeta_3^0 \partial_{\sigma\sigma}\theta e_{\alpha\alpha}(\eta) + 2\mu\zeta_3^0 \partial_{\alpha\beta}\theta e_{\alpha\beta}(\eta) \right) dx + \int_\Omega f_\alpha^1 \eta_\alpha dx \; \forall\, \eta_\alpha \in V_H(\omega). \tag{3.3.47}$$

The left hand side of (3.3.47) is elliptic over $V_H(\omega)$ and the right hand side is linear functional in $(L^2(\omega))^2$. Hence there exists a unique solution $(\zeta_\alpha^1) \in V_H(\omega)$ satisfying (3.3.47).

Taking $v = (\eta_1 - x_3\partial_1\eta_3, \eta_2 - x_3\partial_2\eta_3, \eta_3) \in V$ with η_α, η_3 are independent of x_3, using (3.3.44), the relations

$$\left.\begin{aligned}
&e^0_{\alpha||\beta}(v) = \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) - x_3\partial_{\alpha\beta}\eta_3, \;\; e^1_{\alpha||\beta}(v) = -\eta_3\partial_{\alpha\beta}\theta, \\
&e^{-1}_{\alpha||3}(v) = -\frac{1}{2}\partial_\alpha\eta_3, \; e^0_{\alpha||3}(v) = \frac{1}{2}\partial_\alpha\eta_3, \\
&e^1_{3||3}(v) = \partial_3 v_3 = 0, \; e^0_{3||3}(u) = e^0_{\sigma||\tau}(u) = 0
\end{aligned}\right\} \tag{3.3.48}$$

and $f_3^1 = 0$, the variational equation (3.3.39) becomes

$$\int_\Omega \left\{ A^{\alpha\beta\sigma\tau}(0) e^1_{\sigma||\tau}(u) \left(\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - x_3 \partial_{\alpha\beta}\eta_3 \right) + 4A^{\alpha 3\sigma 3}(0) e^2_{\sigma||3}(u) \left(-\frac{1}{2}\partial_\alpha \eta_3 \right) \right.$$
$$\left. + A^{\alpha\beta 33}(0) e^1_{3||3}(u) \left(\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - x_3 \partial_{\alpha\beta}\eta_3 \right) \right\} dx$$
$$= \int_\omega \left(\int_{-1}^1 f^1_\alpha(x_1, x_2, t) dt \right) \eta_\alpha d\omega - \int_\omega \left(\int_{-1}^1 t f^1_\alpha(x_1, x_2, t) dt \right) \partial_\alpha \eta_3 d\omega \qquad (3.3.49)$$

$\mathbf{Step\ 5}$: $\boldsymbol{\zeta} = (\zeta_1^1, \zeta_2^1, \zeta_3^0)$ satisfies the variational problem (3.3.5)

Using the new order (3.3.43) for the forces and equating the coefficient of ε^2 on both sides of (3.2.18) leads to the problem : Find $u^4 \in H^1(\Omega)$ such that

$$\int_\Omega \{ A^{\alpha\beta\sigma\tau}(0)(e^2_{\sigma||\tau}(u) e^0_{\alpha||\beta}(v) + e^1_{\sigma||\tau}(u) e^1_{\alpha||\beta}(v) + e^0_{\sigma||\tau}(u) e^2_{\alpha||\beta}(v))$$
$$+ B^{\alpha\beta\sigma\tau,1}(0) e^0_{\sigma||\tau}(0) e^0_{\alpha||\beta}(v) + 4A^{\alpha 3\sigma 3}(0)(e^1_{\sigma||3}(u) e^1_{\alpha||3}(v) + e^2_{\sigma||3}(u) e^0_{\alpha||3}(v)$$
$$+ e^{-1}_{\sigma||3}(u) e^3_{\alpha||3}(v) + e^0_{\sigma||3}(u) e^2_{\alpha||3}(v) + e^3_{\sigma||3}(u) e^{-1}_{\alpha||3}(v))$$
$$+ 4B^{\alpha 3\sigma 3,1}(e^{-1}_{\sigma||3}(u) e^1_{\alpha||3}(v) + e^0_{\sigma||3}(u) e^0_{\alpha||3}(v) + e^1_{\sigma||3}(u) e^{-1}_{\alpha||3}(v))$$
$$+ 4B^{\alpha 3\sigma 3,2} e^{-1}_{\sigma||3}(u) e^{-1}_{\alpha||3}(v)$$
$$+ A^{\alpha\beta 33}(0)(e^1_{3||3}(u) e^1_{\alpha||\beta}(v) + e^2_{3||3}(u) e^0_{\alpha||\beta}(v) + e^0_{3||3}(u) e^2_{\alpha||\beta}(v) + e^{-1}_{3||3}(u) e^3_{\alpha||\beta}(v))$$
$$+ B^{\alpha\beta 33,1}(e^0_{3||3}(u) e^0_{\alpha||\beta}(v) + e^{-1}_{3||3}(u) e^1_{\alpha||\beta}(v)$$
$$+ A^{33\sigma\tau}(0) e^3_{\sigma||\tau}(u) \partial_3 v_3 + B^{33\sigma\tau,1} e^1_{\sigma||\tau}(u) \partial_3 v_3 + A^{3333}(0) e^3_{3||3}(u) \partial_3 v_3$$
$$+ B^{3333,1} e^1_{3||3}(u) \partial_3 v_3 + B^{3333,2} e^{-1}_{3||3}(u) \partial_3 v_3 \Big\} dx$$
$$= \int_\Omega f_i^2 v_i dx. \qquad (3.3.50)$$

Using (3.3.12), (3.3.23) and (3.3.42), the above equation becomes

$$\int_\Omega \Big\{ A^{\alpha\beta\sigma\tau}(0)(e^2_{\sigma||\tau}(u)e^0_{\alpha||\beta}(v) + e^1_{\sigma||\tau}(u)e^1_{\alpha||\beta}(v)) + 4B^{\alpha 3\sigma 3,1}e^1_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)$$
$$+ 4A^{\alpha 3\sigma 3}(0)(e^1_{\sigma||3}(u)e^1_{\alpha||3}(v) + e^2_{\sigma||3}(u)e^0_{\alpha||3}(v) + e^3_{\sigma||3}(u)e^{-1}_{\alpha||3}(v))$$
$$+ A^{\alpha\beta 33}(0)(e^1_{3||3}(u)e^1_{\alpha||\beta}(v) + e^2_{3||3}(u)e^0_{\alpha||\beta}(v)) + 4B^{\alpha 3\sigma 3,1}e^1_{\sigma||3}(u)e^{-1}_{\alpha||3}(v)$$
$$+ A^{33\sigma\tau}(0)e^3_{\sigma||\tau}(u)\partial_3 v_3 + B^{33\sigma\tau,1}e^1_{\sigma||\tau}(u)\partial_3 v_3 + A^{3333}(0)e^3_{3||3}(u)\partial_3 v_3$$
$$+B^{3333,1}e^1_{3||3}(u)\partial_3 v_3\Big\}\, dx$$
$$= \int_\Omega f_i^2 v_i dx. \tag{3.3.51}$$

Using (3.2.13) and (3.2.15), it becomes

$$\int_\Omega \Bigg\{ A^{\alpha\beta\sigma\tau}(0)\{e^2_{\sigma||\tau}(u)\frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) + e^1_{\sigma||\tau}(u)(-v_3\partial_{\alpha\beta}\theta)\}$$
$$+ 4A^{\alpha 3\sigma 3}(0)(e^1_{\sigma||3}(u)(v_\gamma\partial_{\alpha\gamma}\theta) + e^2_{\sigma||3}(u)(\frac{1}{2}\partial_\alpha v_3) + e^3_{\sigma||3}(u)(\frac{1}{2}\partial_3 v_\alpha))$$
$$+ A^{\alpha\beta 33}(0)(e^1_{3||3}(u)(-v_3\partial_{\alpha\beta}\theta) + e^2_{3||3}(u)(\frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha))$$
$$+ A^{33\sigma\tau}e^3_{\sigma||\tau}(u)\partial_3 v_3 + B^{33\sigma\tau,1}e^1_{\sigma||\tau}(u)\partial_3 v_3 + A^{3333}(0)e^3_{3||3}(u)\partial_3 v_3$$
$$+B^{3333,1}e^1_{3||3}(u)\partial_3 v_3 + 4B^{\alpha 3\sigma 3,1}e^1_{\sigma||3}(u)\frac{1}{2}\partial_3 v_\alpha\Bigg\}\, dx$$
$$= \int_\Omega f_i^2 v_i dx.$$

As in the proof of step 4, this problem must be considered only for the test functions with vanishing horizontal components (since f^2_α is not yet introduced).

Therefore, if we restrict the test functions of the form $v = (0, 0, \eta_3) \in V_3(\omega)$ with η_3 independent of x_3, this problem reduces to

$$\int_\Omega \Bigg\{ A^{\alpha\beta\sigma\tau}(0)e^1_{\sigma||\tau}(u)(-\eta_3\partial_{\alpha\beta}\theta) + 4A^{\alpha 3\sigma 3}(0)(e^2_{\sigma||3}(u)(\frac{1}{2}\partial_\alpha \eta_3)$$
$$+ A^{\alpha\beta 33}(0)e^1_{3||3}(u)(-\eta_3\partial_{\alpha\beta}\theta)\Big\}\, dx = \int_\omega \left(\int_{-1}^1 f_3^2(x_1, x_2, t)dt\right)\eta_3 d\omega. \tag{3.3.52}$$

Adding (3.3.49) and (3.3.52), we have

$$\begin{aligned}
&\int_{\Omega} \Big\{ A^{\alpha\beta\sigma\tau}(0) e^1_{\sigma||\tau}(u) \left(\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - x_3 \partial_{\alpha\beta}\eta_3 - \eta_3 \partial_{\alpha\beta}\theta \right) \\
&\quad + A^{\alpha\beta 33}(0) e^1_{3||3}(u) \left(\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - x_3 \partial_{\alpha\beta}\eta_3 - \eta_3 \partial_{\alpha\beta}\theta \right) \Big\} dx \\
&\quad = \int_\omega \left(\int_{-1}^{1} f_3^2(x_1, x_2, t) dt \right) \eta_3 d\omega + \int_\omega \left(\int_{-1}^{1} f_\alpha^1(x_1, x_2, t) dt \right) \eta_\alpha d\omega \\
&\quad\quad - \int_\omega \left(\int_{-1}^{1} t f_\alpha^1(x_1, x_2, t) dt \right) \partial_\alpha \eta_3 d\omega.
\end{aligned} \tag{3.3.53}$$

Note that

$$\begin{aligned}
\widetilde{e}_{\alpha\beta}(v) &= \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - v_3 \partial_{\alpha\beta}\theta \\
&= \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - x_3 \partial_{\alpha\beta}\eta_3 - \eta_3 \partial_{\alpha\beta}\theta \ \text{ when } \ v = (\eta_\alpha - x_3 \partial_\alpha \eta_3, \ \eta_3).
\end{aligned}$$

Using (3.2.3), (3.3.30) and (3.3.46) we have

$$\begin{aligned}
\text{LHS of (3.3.53)} &= \int_\Omega \big(\big(\lambda \delta^{\alpha\beta} \delta^{\sigma\tau} e^1_{\sigma||\tau}(u) + \lambda \delta^{\alpha\beta} e^1_{3||3}(u) \big) \\
&\quad + \mu (\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\tau}) e^1_{\sigma||\tau}(u) \big) \, \widetilde{e}_{\alpha\beta}(v) dx \\
&= \int_\Omega \big(\lambda \delta^{\alpha\beta} e^1_{p||p}(u) + 2\mu e^1_{\alpha||\beta}(u) \big) \, \widetilde{e}_{\alpha\beta}(v) dx \\
&= -\int_\omega m^{\alpha\beta}(w) \partial_{\alpha\beta}\eta_3 d\omega - \int_\omega \widetilde{n}^{\alpha\beta}(w) \eta_3 \partial_{\alpha\beta}\theta d\omega + \int_\omega \widetilde{n}^{\alpha\beta}(w) \partial_\beta \eta_\alpha d\omega
\end{aligned}$$

Hence $\boldsymbol{\zeta} = (\zeta_1^1, \zeta_2^1, \zeta_3^0)$ satisfies (3.3.5).

□

Conclusions: The above asymptotic analysis of the three-dimensional problem (3.1.1) showed that with the assumption (3.1.4), if the expansion of the displacement field $u(\varepsilon)$ is as in (3.2.5), then the asymptotic orders of force must be:

$$f_\alpha(\varepsilon) = \varepsilon f_\alpha^1 \in L^2(\Omega), \ f_3(\varepsilon) = \varepsilon^2 f_3^2 \in L^2(\Omega).$$

This implies that the asymptotic order of the components of the displacement field is:

$$u_\alpha(\varepsilon) = \varepsilon u_\alpha^1,\ u_3(\varepsilon) = u_3^0.$$

However these scalings on the data $f(\varepsilon)$ and the unknowns $u(\varepsilon)$ deviate from those of (3.1.3) and (3.1.5) by a multiplicative factor of ε, which can be explained by the linearity of the problem (3.2.18) and the choice of the expansion (3.2.5).

CHAPTER 4

Two-dimensional Approximation of Piezoelectric Shallow Shells with Variable Thickness

4.1 Introduction

In this chapter, we consider the boundary value problem for a thin piezoelectric shallow shell with *variable* thickness. As the thickness of the shell is small, the question is: what is the two dimensional approximation of this model?

Thus our aim in this chapter is to identify the two dimensional model which is the approximation of the three dimensional thin piezoelectric shell with *variable* thickness.

We first pose the problem in variational form and transfer the problem, by making suitable scalings on the unknowns and data, to a domain which is independent of the thickness parameter. Then we derive a Korn's type inequality which is needed to show that the solutions are bounded in a suitable Hilbert space which would imply a weakly convergence subsequence (of solutions) and choosing suitable test functions we pass to the limit in the variational formulation to obtain the limiting model. We then show that the limiting model has a unique solution and the solutions converges strongly.

This chapter is organized as follows. In section 4.2, we describe the three dimensional problem. In section 4.3, we transform the problem to scaled domain and in section 4.4, we discuss the technical preliminaries. In section 4.5, we study the limit problem.

4.2 The Three-dimensional Problem

Let $\omega \subset I\!R^2$ be a bounded domain with a Lipschitz continuous boundary γ and let ω lie locally on one side of γ. Let $\gamma_0, \gamma_h \subset \partial\omega$ with meas$(\gamma_0) > 0$ and meas$(\gamma_h) > 0$. Let

$\gamma_1 = \partial\omega\backslash\gamma_0$ and $\gamma_s = \partial\omega\backslash\gamma_h$. For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon),\ \ \Gamma^{\pm,\varepsilon} = \omega \times \{\pm\varepsilon\},\ \ \Gamma_0^\varepsilon = \gamma_0 \times (-\varepsilon, \varepsilon),\ \ \Gamma_1^\varepsilon = \gamma_1 \times (-\varepsilon, \varepsilon),$$

$$\Gamma_N^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma^{\pm,\varepsilon},\ \ \Gamma_h^\varepsilon = \gamma_h \times (-\varepsilon, \varepsilon),\ \ \Gamma_s^\varepsilon = \gamma_s \times (-\varepsilon, \varepsilon).$$

Let $x^\varepsilon = (x_1, x_2, x_3^\varepsilon)$ be a generic point on $\bar{\Omega}^\varepsilon$ and let $\partial_\alpha = \partial_\alpha^\varepsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\varepsilon = \frac{\partial}{\partial x_3^\varepsilon}$.

We assume that for each ε, we are given a function $\theta^\varepsilon : \bar{\omega} \to I\!R$ of class $\mathcal{C}^3$. We then define the map $\phi^\varepsilon : \bar{\omega} \to I\!R^3$ by

$$\phi^\varepsilon(x_1, x_2) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) \text{ for all } (x_1, x_2) \in \bar{\omega}. \tag{4.2.1}$$

At each point of the surface $S^\varepsilon = \phi^\varepsilon(\bar{\omega})$, we define the normal vector

$$a^\varepsilon = (|\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2 + 1)^{-\frac{1}{2}}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1).$$

The variable thickness of the shell is governed by a function $e \in W^{2,\infty}(\omega)$ such that there exists a constant e_0 such that

$$0 < e_0 < e(x_1, x_2) \text{ for all } (x_1, x_2) \in \bar{\omega}.$$

For each $\varepsilon > 0$, we define the mapping $\Phi^\varepsilon : \bar{\Omega}^\varepsilon \to I\!R^3$ by

$$\Phi^\varepsilon(x^\varepsilon) = \phi^\varepsilon(x_1, x_2) + x_3^\varepsilon e(x_1, x_2) a^\varepsilon(x_1, x_2) \text{ for all } x^\varepsilon \in \bar{\Omega}^\varepsilon. \tag{4.2.2}$$

Hence at the point $\Phi^\varepsilon(x^\varepsilon)$ the thickness is $2\varepsilon e(x_1, x_2)$.

The set $\bar{\hat{\Omega}}^\varepsilon = \Phi^\varepsilon(\bar{\Omega}^\varepsilon)$ is the reference configuration of the shell and we denote a generic point of the shell by $\hat{x}^\varepsilon$.

For $0 < \varepsilon \leq \varepsilon_0$, we define the sets

$$\hat{\Gamma}^{\pm,\varepsilon} = \Phi^\varepsilon(\Gamma^{\pm,\varepsilon}),\ \ \hat{\Gamma}_0^\varepsilon = \Phi^\varepsilon(\Gamma_0^\varepsilon),\ \ \hat{\Gamma}_1^\varepsilon = \Phi(\Gamma_1^\varepsilon),\ \ \hat{\Gamma}_N^\varepsilon = \hat{\Gamma}_1^{\ \varepsilon} \cup \hat{\Gamma}^{\pm\varepsilon},$$

$$\hat{\Gamma}^{\varepsilon}_{h} = \Phi(\Gamma^{\varepsilon}_{h}),\ \ \hat{\Gamma}^{\varepsilon}_{s} = \Phi(\Gamma^{\varepsilon}_{s}),\ \ \hat{\Gamma}^{\varepsilon}_{hD} = \hat{\Gamma}^{+\varepsilon} \cup \hat{\Gamma}^{-\varepsilon} \cup \hat{\Gamma}^{\varepsilon}_{h}.$$

We define vectors g_i^{ε} and $g^{i,\varepsilon}$ by the relations

$$g_i^{\varepsilon} = \partial_i^{\varepsilon} \Phi^{\varepsilon} \text{ and } g^{j,\varepsilon} \cdot g_i^{\varepsilon} = \delta_i^j.$$

which form the covariant and contravariant basis respectively of the tangent plane of $\Phi^{\varepsilon}(\bar{\Omega}^{\varepsilon})$ at $\Phi^{\varepsilon}(x^{\varepsilon})$. The covariant and contravariant metric tensors are given, respectively, by

$$g_{ij}^{\varepsilon} = g_i^{\varepsilon} \cdot g_j^{\varepsilon} \ \text{ and } \ g^{ij,\varepsilon} = g^{i,\varepsilon} \cdot g^{j,\varepsilon}.$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\varepsilon} = g^{p,\varepsilon} \cdot \partial_j^{\varepsilon} g_i^{\varepsilon}.$$

Note however that when the set Ω^{ε} is of the special form $\Omega^{\varepsilon} = \omega \times (-\varepsilon, \varepsilon)$ and the mapping Φ^{ε} is of the form (4.2.2), the following relations hold,

$$\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0.$$

The volume element is given by $\sqrt{g^{\varepsilon}} dx^{\varepsilon}$ where

$$g^{\varepsilon} = det(g_{ij}^{\varepsilon}).$$

We assume that the material is mechanically isotropic so that the elasticity tensor $\hat{A}^{ijkl,\varepsilon}$ is given by

$$\hat{A}^{ijkl,\varepsilon} = \lambda \delta^{ij} \delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{4.2.3}$$

where λ and μ are Lamé constants. Clearly this tensor satisfy the symmetry relations

$$\hat{A}^{ijkl,\varepsilon} = \hat{A}^{jikl,\varepsilon} = \hat{A}^{klij,\varepsilon} \tag{4.2.4}$$

and the inequality

$$\hat{A}^{ijkl,\varepsilon} t_{ij} t_{kl} \geq C \sum_{i,j} |t_{ij}|^2 \tag{4.2.5}$$

for all symmetric tensor (t_{ij}).

Let $\hat{P}^{ijk,\varepsilon}$ and $\hat{\in}^{ij,\varepsilon}$ denote the piezoelectric and dielectric tensors respectively. We assume that they are symmetric and there exists $C > 0$ such that

$$\hat{\in}^{ij,\varepsilon} t_i t_j \geq C \sum_i |t_i|^2 \tag{4.2.6}$$

for all $(t_i) \in I\!R^3$.

Let $\hat{f}^\varepsilon$ and $\hat{\varphi}_0^\varepsilon$ be the applied body force and electric potential.

Then the boundary value problem consists of finding $(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon)$ such that

$$\left.\begin{aligned} -div\hat{\sigma}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon) = \hat{f}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \\ \hat{\sigma}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon)\nu = 0 \ \text{ on } \hat{\Gamma}^\varepsilon_N, \\ \hat{u}^\varepsilon = 0 \ \text{ on } \hat{\Gamma}^\varepsilon_0, \end{aligned}\right\} \tag{4.2.7}$$

$$\left.\begin{aligned} div\hat{D}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon) = 0 \ \text{ in } \hat{\Omega}^\varepsilon, \\ \hat{D}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon)\nu = 0 \ \text{ on } \hat{\Gamma}^\varepsilon_s, \\ \hat{\varphi}^\varepsilon = \hat{\varphi}_0^\varepsilon \text{ on } \hat{\Gamma}^\varepsilon_{hD}. \end{aligned}\right\} \tag{4.2.8}$$

where

$$\hat{\sigma}^\varepsilon_{ij} = \hat{A}^{ijkl,\varepsilon}\hat{e}^\varepsilon_{ij} - \hat{P}^{kij,\varepsilon}\hat{E}^\varepsilon_k, \tag{4.2.9}$$

$$\hat{D}^\varepsilon_k = \hat{P}^{kij,\varepsilon}\hat{e}^\varepsilon_{ij} + \hat{\in}^{kl,\varepsilon}\hat{E}^\varepsilon_l, \tag{4.2.10}$$

$$\hat{e}^\varepsilon_{ij}(\hat{u}^\varepsilon) = \frac{1}{2}(\hat{\partial}^\varepsilon_i \hat{u}^\varepsilon_j + \hat{\partial}^\varepsilon_j \hat{u}^\varepsilon_i), \ \ \hat{\partial}^\varepsilon_i = \frac{\partial}{\partial \hat{x}^\varepsilon_i} \ \text{ and } \ \hat{E}^\varepsilon_k(\hat{\varphi}^\varepsilon) = -\hat{\bigtriangledown}^\varepsilon(\hat{\varphi}^\varepsilon).$$

Let $\overline{\hat{\varphi}^\varepsilon} = \hat{\varphi}^\varepsilon - \hat{\varphi}^\varepsilon_0$, where $\hat{\varphi}^\varepsilon_0$ is a trace lifting in $H^1(\hat{\Omega}^\varepsilon)$ of the boundary potential.

We define the spaces

$$\hat{V}^\varepsilon = \{\hat{v}^\varepsilon \in (H^1(\hat{\Omega}^\varepsilon))^3, \hat{v}|_{\hat{\Gamma}^\varepsilon_0} = 0\}, \tag{4.2.11}$$

$$\hat{\Psi}^\varepsilon = \{\hat{\psi}^\varepsilon \in H^1(\hat{\Omega}^\varepsilon), \hat{\psi}|_{\hat{\Gamma}^\varepsilon_{hD}} = 0\}. \tag{4.2.12}$$

The variational form of the system (4.2.7)-(4.2.8) is to find $(\hat{u}^\varepsilon, \overline{\hat{\varphi}}^\varepsilon) \in \hat{V}^\varepsilon \times \hat{\Psi}^\varepsilon$ such that

$$\hat{a}^\varepsilon((\hat{u}^\varepsilon, \overline{\hat{\varphi}}^\varepsilon), (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon)) = \hat{l}^\varepsilon(\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) \text{ for all } (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) \in \hat{V}^\varepsilon \times \hat{\Psi}^\varepsilon \tag{4.2.13}$$

where

$$\begin{aligned}\hat{a}^\varepsilon((\hat{u}^\varepsilon, \overline{\hat{\varphi}}^\varepsilon), (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon)) = &\int_{\hat{\Omega}^\varepsilon} \hat{A}^{ijkl,\varepsilon} \hat{e}^\varepsilon_{kl}(\hat{u}^\varepsilon) \hat{e}^\varepsilon_{ij}(\hat{v}^\varepsilon) d\hat{x}^\varepsilon + \int_{\hat{\Omega}^\varepsilon} \hat{\in}^{ij,\varepsilon} \hat{\partial}^\varepsilon_i \overline{\hat{\varphi}}^\varepsilon \hat{\partial}^\varepsilon_j \hat{\psi}^\varepsilon d\hat{x}^\varepsilon \\ &+ \int_{\hat{\Omega}^\varepsilon} \hat{P}^{mij,\varepsilon} \left(\hat{\partial}^\varepsilon_m \overline{\hat{\varphi}}^\varepsilon \hat{e}^\varepsilon_{ij}(\hat{v}^\varepsilon) - \hat{\partial}^\varepsilon_m \hat{\psi}^\varepsilon \hat{e}^\varepsilon_{ij}(\hat{u}^\varepsilon) \right) d\hat{x}^\varepsilon \end{aligned} \tag{4.2.14}$$

$$\hat{l}^\varepsilon(\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} \hat{f}^\varepsilon \cdot \hat{v}^\varepsilon d\hat{x}^\varepsilon - \int_{\hat{\Omega}^\varepsilon} \hat{\in}^{ij,\varepsilon} \hat{\partial}^\varepsilon_i \hat{\varphi}^\varepsilon_0 \hat{\partial}^\varepsilon_j \hat{\psi}^\varepsilon dx^\varepsilon - \int_{\hat{\Omega}^\varepsilon} \hat{P}^{mij,\varepsilon} \hat{\varphi}^\varepsilon_0 \hat{e}_{ij}(\hat{v}^\varepsilon) dx^\varepsilon \tag{4.2.15}$$

Since the mappings $\Phi^\varepsilon : \overline{\Omega}^\varepsilon \to \overline{\hat{\Omega}}^\varepsilon$ are assumed to be $\mathcal{C}^1$ diffeomorphism, the correspondence that associates with every vector $\hat{v}^\varepsilon = (\hat{v}^\varepsilon_i) \in \hat{V}^\varepsilon$ (note that $(\hat{v}^\varepsilon_i)$ are the components of the vector $\hat{v}^\varepsilon = \hat{v}^\varepsilon_i \hat{e}^i$, where $(\hat{e}^i)^3_{i=1}$ is the standard basis in $I\!R^3$) the vector $v^\varepsilon = (v^\varepsilon_i)$ defined by

$$\hat{v}^\varepsilon_i(\hat{x}^\varepsilon)\hat{e}^i = v^\varepsilon_i(x^\varepsilon) g^i(x^\varepsilon)$$

induces a bijection between the spaces $\hat{V}^\varepsilon$ and V^ε, where

$$V^\varepsilon = \{v^\varepsilon \in (H^1(\Omega^\varepsilon))^3 | v^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_0\}. \tag{4.2.16}$$

Then we have (cf. Ciarlet (2000))

$$\hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_i(\hat{x}^\varepsilon) = (\partial^\varepsilon_l v^\varepsilon_k - \Gamma^{q,\varepsilon}_{lk} v^\varepsilon_q)(g^{k,\varepsilon})_i (g^{l,\varepsilon})_j, \tag{4.2.17}$$

$$\hat{e}_{ij}(\hat{v}^\varepsilon)(\hat{x}^\varepsilon) = e^\varepsilon_{k||l}(v^\varepsilon)(g^{k,\varepsilon})_i (g^{l,\varepsilon})_j, \tag{4.2.18}$$

where

$$e^\varepsilon_{i||j}(v^\varepsilon) = \frac{1}{2}(\partial^\varepsilon_i v^\varepsilon_j + \partial^\varepsilon_j v^\varepsilon_i) - \Gamma^{p,\varepsilon}_{ij} v^\varepsilon_p. \tag{4.2.19}$$

Also with any scalar function $\hat{\varphi}^\varepsilon \in \hat{\Psi}^\varepsilon$, the correspondence $\hat{\varphi}^\varepsilon(\hat{x}^\varepsilon) = \varphi^\varepsilon(x^\varepsilon)$ induces a bijection between the spaces $\hat{\Psi}^\varepsilon$ and Ψ^ε where

$$\Psi^\varepsilon = \{\psi^\varepsilon \in H^1(\Omega^\varepsilon) | \psi^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{hD}\}. \tag{4.2.20}$$

Then

$$\hat{\partial}_j \hat{\varphi}^\varepsilon = \hat{\partial}_j \varphi^\varepsilon(x^\varepsilon) = \hat{\partial}_j \varphi^\varepsilon((\Phi^\varepsilon)^{-1}(\hat{x}^\varepsilon)) = \partial_l \varphi^\varepsilon(x^\varepsilon)(g^l(x^\varepsilon))_j. \tag{4.2.21}$$

Then the variational problem consists of finding $(u^\varepsilon, \overline{\varphi}^\varepsilon)$ such that

$$a^\varepsilon((u^\varepsilon, \overline{\varphi}^\varepsilon), (v^\varepsilon, \psi^\varepsilon)) = l^\varepsilon(v^\varepsilon, \psi^\varepsilon) \text{ for all } (v^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times \Psi^\varepsilon \tag{4.2.22}$$

where

$$\begin{aligned} a^\varepsilon((u^\varepsilon, \varphi^\varepsilon), (v^\varepsilon, \psi^\varepsilon)) = & \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(v^\varepsilon) e^\varepsilon_{i||j}(v^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} \in^{ij,\varepsilon} \partial^\varepsilon_i \overline{\varphi}^\varepsilon \partial^\varepsilon_j \psi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \\ & + \int_{\Omega^\varepsilon} P^{mij,\varepsilon} \left(\partial^\varepsilon_m \overline{\varphi}^\varepsilon e^\varepsilon_{i||j}(v^\varepsilon) - \partial^\varepsilon_m \psi^\varepsilon e^\varepsilon_{i||j}(u^\varepsilon) \right) \sqrt{g^\varepsilon} dx^\varepsilon, \end{aligned} \tag{4.2.23}$$

$$\begin{aligned} l^\varepsilon(v^\varepsilon, \psi^\varepsilon) = & \int_{\Omega^\varepsilon} f^\varepsilon \cdot v^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon - \int_{\Omega^\varepsilon} \in^{ij,\varepsilon} \partial^\varepsilon_i \varphi^\varepsilon_0 \partial^\varepsilon_j \psi^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \\ & - \int_{\Omega^\varepsilon} P^{mij,\varepsilon} \left(\partial^\varepsilon_m \varphi^\varepsilon_0 e^\varepsilon_{i||j}(v^\varepsilon) - \partial^\varepsilon_m \psi^\varepsilon e^\varepsilon_{i||j}(u^\varepsilon) \right) \sqrt{g^\varepsilon} dx^\varepsilon, \end{aligned} \tag{4.2.24}$$

$$A^{ijkl,\varepsilon} = \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}), \tag{4.2.25}$$

$$P^{pqr,\varepsilon} = \hat{P}^{ijk,\varepsilon}.(g^{p,\varepsilon})_i (g^{q,\varepsilon})_j (g^{r,\varepsilon})_k, \tag{4.2.26}$$

$$\in^{pq,\varepsilon} = \hat{\in}^{ij,\varepsilon} (g^{p,\varepsilon})_i (g^{q,\varepsilon})_j, \tag{4.2.27}$$

It can be shown that there exists a constant $C > 0$ such that for all symmetric tensors (t_{ij})

$$A^{ijkl,\varepsilon} t_{kl} t_{ij} \geq C \sum_{i,j=1}^{3} (t_{ij})^2. \tag{4.2.28}$$

Using (4.2.6) and that $(g^{j,\varepsilon})$ forms contravariant basis, it follows that for any vector

$(t_i) \in I\!R^3$

$$\in^{kl,\varepsilon} t_k t_l \geq C \sum_{j=1}^{3} t_j^2. \tag{4.2.29}$$

Moreover from the symmetry of $\hat{A}^{ijkl,\varepsilon},\ \hat{P}^{ijk,\varepsilon},\ \hat{\in}^{ij,\varepsilon}$ we have the symmetries

$$A^{ijkl,\varepsilon} = A^{klij,\varepsilon} = A^{jikl,\varepsilon}, \in^{kl,\varepsilon} = \in^{lk,\varepsilon}, P^{ijk,\varepsilon} = P^{kij,\varepsilon}. \tag{4.2.30}$$

Using (4.2.28) and (4.2.29) we have

$$\begin{aligned} a^\varepsilon((u^\varepsilon, \varphi^\varepsilon), (u^\varepsilon, \varphi^\varepsilon)) &= \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k||l}(u^\varepsilon) e^\varepsilon_{i||j}(u^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} \in^{ij,\varepsilon} \partial^\varepsilon_i \overline{\varphi}^\varepsilon \partial^\varepsilon_j \overline{\varphi}^\varepsilon \sqrt{g^\varepsilon} dx^\varepsilon \\ &\geq C(||u^\varepsilon||^2_{1,\Omega^\varepsilon} + ||\overline{\varphi}^\varepsilon||^2_{1,\Omega^\varepsilon}). \end{aligned} \tag{4.2.31}$$

Hence the bilinear form $a^\varepsilon(\cdot,\cdot)$associated with the left-hand side of (4.2.22) is elliptic and the linear form $l^\varepsilon(\cdot)$ is continuous. Hence by Lax-Milgram theorem there exists a unique $(u^\varepsilon, \varphi^\varepsilon)$ such that

$$a^\varepsilon((u^\varepsilon, \varphi^\varepsilon), (v^\varepsilon, \psi^\varepsilon)) = l^\varepsilon(v^\varepsilon, \psi^\varepsilon) \text{ for all } (v^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times \Psi^\varepsilon. \tag{4.2.32}$$

4.3 The Scaled Problem

We now perform a change of variable so that the domain no longer depends on ε. With $x = (x_1, x_2, x_3) \in \bar{\Omega}$, we associate $x^\varepsilon = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon$. Let

$$\Gamma_0 = \gamma_0 \times (-1,1),\ \ \Gamma_1 = \gamma_1 \times (-1,1),\ \ \Gamma^\pm = \omega \times \{\pm 1\},\ \ \Gamma_h = \gamma_h \times (-1,1),$$

$$\Gamma_s = \gamma_s \times (-1,1),\ \ \Gamma_N = \Gamma_1 \cup \Gamma^+ \cup \Gamma^-,\ \ \Gamma_{hD} = \Gamma^+ \cup \Gamma^- \cup \Gamma_h.$$

With the functions $\Gamma^{p,\varepsilon}, g^\varepsilon, A^{ijkl,\varepsilon}, P^{ijk,\varepsilon}, \in^{ij,\varepsilon}\colon \bar{\Omega}^\varepsilon \to I\!R$, we associate the functions $\Gamma^p(\varepsilon),\ g(\varepsilon), A^{ijkl}(\varepsilon),\ P^{ijk}(\varepsilon),\ \in^{ij}(\varepsilon) : \bar{\Omega} \to I\!R$ defined by

$$\Gamma^p(\varepsilon)(x) = \Gamma^{p,\varepsilon}(x^\varepsilon),\ \ g(\varepsilon)(x) = g^\varepsilon(x^\varepsilon),\ \ A^{ijkl}(\varepsilon)(x) = A^{ijkl,\varepsilon}(x^\varepsilon), \tag{4.3.1}$$

$$P^{ijk}(\varepsilon)(x) = P^{ijk,\varepsilon}(x^\varepsilon), \quad \in^{ij}(\varepsilon)(x) = \in^{ij,\varepsilon}(x^\varepsilon). \tag{4.3.2}$$

Assumption: We assume that the shell is a shallow shell; i.e., there exists a function $\theta \in \mathcal{C}^3(\bar{\omega})$ such that $\theta^\varepsilon = \varepsilon\theta(x_1, x_2)$;

$$i.e., \quad \phi^\varepsilon(x_1, x_2) = (x_1, x_2, \varepsilon\theta(x_1, x_2)) \text{ for all } (x_1, x_2) \in \bar{\omega}. \tag{4.3.3}$$

In this case, we make the following scalings on the data and unknowns.

$$f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 f_\alpha(\varepsilon)(x), \quad f_3^\varepsilon(x^\varepsilon) = \varepsilon^3 f_3(x), \quad \varphi_0^\varepsilon(x^\varepsilon) = \varepsilon^3\varphi_0(\varepsilon), \tag{4.3.4}$$

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), \;\; v_\alpha(x^\varepsilon) = \varepsilon^2 v_\alpha(x), \quad u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), \quad v_3(x^\varepsilon) = \varepsilon v_3(x), \tag{4.3.5}$$

$$\overline{\varphi}^\varepsilon(x^\varepsilon) = \overline{\varphi}(\varepsilon)(x), \tag{4.3.6}$$

$$E_\alpha(\varepsilon)(\varphi(\varepsilon)) = \varepsilon^{-2}E_\alpha^\varepsilon(\varphi^\varepsilon) = -\varepsilon\partial_\alpha\varphi(\varepsilon), \quad E_3(\varepsilon)(\varphi(\varepsilon)) = \varepsilon^{-2}E_3^\varepsilon(\varphi^\varepsilon) = -\varepsilon\partial_3\varphi(\varepsilon), \tag{4.3.7}$$

$$D_i(\varepsilon)(u(\varepsilon), \varphi(\varepsilon)) = \varepsilon^{-2}D_i^\varepsilon(u^\varepsilon, \varphi^\varepsilon). \tag{4.3.8}$$

With the tensors $e_{i||j}^\varepsilon$, we associate the tensors $e_{i||j}(\varepsilon)$ through the relation

$$e_{i||j}^\varepsilon(v^\varepsilon)(x^\varepsilon) = \varepsilon^2 e_{i||j}(\varepsilon; v)(x). \tag{4.3.9}$$

We define the spaces

$$V(\Omega) = \{v \in (H^1(\Omega))^3, v|_{\Gamma_0} = 0\}, \tag{4.3.10}$$

$$\Psi(\Omega) = \{\psi \in H^1(\Omega), \psi|_{\Gamma_{hD}} = 0\}. \tag{4.3.11}$$

Then the variational problem (4.2.22) becomes: find $(u(\varepsilon), \varphi(\varepsilon)) \in V(\Omega) \times \Psi(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon, u(\varepsilon))e_{i||j}(\varepsilon, v)\sqrt{g(\varepsilon)}dx + \int_\Omega \in^{33}(\varepsilon)\partial_3\overline{\varphi}(\varepsilon)\partial_3\psi\sqrt{g(\varepsilon)}dx \\
&+ \int_\Omega P^{3kl}\left[\partial_3\overline{\varphi}(\varepsilon)e_{k||l}(\varepsilon, v) - \partial_3\psi e_{k||l}(\varepsilon, u(\varepsilon))\right]\sqrt{g(\varepsilon)}dx \\
&+\varepsilon\int_\Omega \in^{3\alpha}(\varepsilon)\left[\partial_\alpha\overline{\varphi}(\varepsilon)\partial_3\psi + \partial_3\overline{\varphi}(\varepsilon)\partial_\alpha\psi)\right]\sqrt{g(\varepsilon)}dx \\
&+\varepsilon\int_\Omega \left[P^{\alpha kl}(\varepsilon)\partial_\alpha\overline{\varphi}(\varepsilon)e_{k||l}(\varepsilon, u(\varepsilon)) - \partial_\alpha\psi e_{k||l}(\varepsilon, v)\right]\sqrt{g(\varepsilon)}dx \\
&+\varepsilon^2\int_\Omega \in^{\alpha\beta}(\varepsilon)\partial_\alpha\overline{\varphi}(\varepsilon)\partial_\beta\psi\sqrt{g(\varepsilon)}dx \\
&= \int_\Omega f \cdot v\sqrt{g(\varepsilon)}dx - \int_\Omega \in^{33}(\varepsilon)\partial_3\varphi_0\partial_3\psi\sqrt{g(\varepsilon)}dx \\
&-\varepsilon\int_\Omega \left[\in^{3\alpha}(\varepsilon)(\partial_\alpha\varphi_0\partial_3\psi + \partial_3\varphi_0\partial_\alpha\psi)\right]\sqrt{g(\varepsilon)}dx - \varepsilon^2\int_\Omega \in^{\alpha\beta}(\varepsilon)\partial_\alpha\varphi_0\partial_\beta\psi\sqrt{g(\varepsilon)}dx \\
&- \int_\Omega P^{3kl}\partial_3\varphi_0 e_{k||l}(\varepsilon, v)\sqrt{g(\varepsilon)}dx - \varepsilon\int_\Omega P^{\alpha ij}\partial_3\varphi_0 e_{i||j}(\varepsilon, v)\sqrt{g(\varepsilon)}dx. \qquad (4.3.12)
\end{aligned}$$

4.4 Technical Preliminaries

The following lemmas are crucial; they play an important role in the proof of the convergence of the scaled unknowns as $\varepsilon \to 0$. In the sequel, we denote by $C_1, C_2, \ldots, C_n$ various constants whose values do not depend on ε but may depend on θ.

Lemma 4.4.1. *The functions* $e_{i||j}(\varepsilon, v)$ *defined in* (4.3.9) *are of the form*

$$e_{\alpha||\beta}(\varepsilon; v) = \tilde{e}_{\alpha\beta}(v) + \varepsilon^2 e^\sharp_{\alpha||\beta}(\varepsilon; v), \qquad (4.4.1)$$

$$e_{\alpha||3}(\varepsilon; v) = \frac{1}{\varepsilon}\left\{\tilde{e}_{\alpha3}(v) + \varepsilon^2 e^\sharp_{\alpha||3}(\varepsilon; v)\right\}, \qquad (4.4.2)$$

$$e_{3||3}(\varepsilon; v) = \frac{1}{\varepsilon^2}\tilde{e}_{33}(v), \qquad (4.4.3)$$

where

$$\tilde{e}_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \frac{v_3}{e}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}e), \qquad (4.4.4)$$

$$\tilde{e}_{\alpha3}(v) = \frac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha), \qquad (4.4.5)$$

$$\tilde{e}_{33}(v) = \partial_3 v_3. \qquad (4.4.6)$$

and there exists constant C_1 such that

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{\alpha,j} ||e^{\sharp}_{\alpha,j}(\varepsilon; v)||_{0,\Omega} \leq C_1||v||_{1,\Omega} \quad \textit{for all} \quad v \in V. \tag{4.4.7}$$

Also there exist constants C_2, C_3 and C_4 such that

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{x\in\bar{\Omega}} |g(x) - e^2| \leq C_2\varepsilon^2, \tag{4.4.8}$$

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{x\in\bar{\Omega}} |A^{ijkl}(\varepsilon) - A^{ijkl}(0)| \leq C_3\varepsilon^2, \tag{4.4.9}$$

where

$$A^{\alpha\beta\sigma\tau}(0) = \lambda\delta^{\alpha\beta}\delta^{\sigma\tau} + \mu(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma}), \tag{4.4.10}$$

$$A^{\alpha\beta\sigma3}(0) = 0, \ \ A^{\alpha\beta33}(0) = \frac{1}{e^2}\lambda\delta^{\alpha\beta}, \ \ A^{\alpha3\sigma3}(0) = \frac{1}{e^2}\mu\delta^{\alpha\sigma}, \tag{4.4.11}$$

$$A^{\alpha333}(0) = 0, \ \ A^{3333} = \frac{1}{e^4}(\lambda + 2\mu), \tag{4.4.12}$$

$$A^{ijkl}(\varepsilon)t_{kl}t_{ij} \geq C_4 t_{ij}t_{ij}, \tag{4.4.13}$$

for $0 < \varepsilon \leq \varepsilon_0$, for all $x \in \bar{\Omega}$, and for all symmetric tensors (t_{ij}).

Proof. A simple computation using (4.3.3) shows that

$$g_\alpha(\varepsilon) = \begin{pmatrix} \delta_{\alpha1} - \varepsilon^2 x_3[e\partial_{\alpha1}\theta + \partial_1\theta\partial_\alpha e] + O(\varepsilon^2) \\ \delta_{\alpha2} - \varepsilon^2 x_3[\partial_{\alpha2}\theta + \partial_2\theta\partial_\alpha e] + O(\varepsilon^2) \\ \varepsilon[\partial_\alpha\theta + x_3\partial_\alpha e] + O(\varepsilon^4) \end{pmatrix}, \tag{4.4.14}$$

$$g_3(\varepsilon) = \begin{pmatrix} -\varepsilon\partial_1\theta + O(\varepsilon^3) \\ -\varepsilon\partial_2\theta + O(\varepsilon^3) \\ e + O(\varepsilon^2) \end{pmatrix}. \tag{4.4.15}$$

Hence

$$g_{\alpha\beta}(\varepsilon) = \delta_{\alpha\beta} + \varepsilon^2[\partial_\alpha\theta\partial_\beta\theta - 2x_3(e\partial_{\alpha\beta}\theta + \partial_\alpha\theta\partial_\beta)e] + O(\varepsilon^4), \tag{4.4.16}$$

$$g_{\alpha 3}(\varepsilon) = O(\varepsilon), \;\; g_{33}(\varepsilon) = e^2 + O(\varepsilon^2), \tag{4.4.17}$$

$$\Gamma^{\sigma}_{\alpha\beta}(\varepsilon) = O(\varepsilon^2), \;\; \Gamma^{3}_{\alpha\beta}(\varepsilon) = \frac{\varepsilon}{e}[\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}e] + O(\varepsilon^3), \;\; \Gamma^{\sigma}_{\alpha 3} = O(\varepsilon). \tag{4.4.18}$$

The announced results follows from the above relations. $\square$

Lemma 4.4.2. *(Duvaut and Lions (1972)) Let Ω be a Lipschitz continuous domain in $I\!R^n$, and let v be a distribution in $I\!R^n$. Then*

$$v \in H^{-1}(\Omega) \;\; \partial_i v \in H^{-1}(\Omega) \; for \; all \; 1 \leq i \leq n \Rightarrow v \in L^2(\Omega)$$

Theorem 4.4.3. *(Hörmander (1983)). If $P(x,\xi) = \sum_{ij} a_{ij}\xi_i\xi_j$ where a_{ij} are Lipschitz-continuous in a neighbourhood of zero, $P(x,\xi)$ is elliptic and if $u \in H^1(\omega)$ satisfies $|P(x,D)u| \leq C \sum_{|\alpha|\leq 1} |D^\alpha u|$ then $u = 0$ in ω if u vanishes in a neighbourhood of a point in ω.*

Lemma 4.4.4. *Let $\theta \in \mathcal{C}^3(\overline{\omega})$ be a given function and let the functions $\tilde{e}_{ij}(v)$ be defined as in (4.4.4)-(4.4.6). Then there exists a constant C_5 such that*

$$||v||_{1,\Omega} \leq C_5 \left\{ \sum_{i,j} ||\tilde{e}_{ij}(v)||^2 \right\}^{\frac{1}{2}} \tag{4.4.19}$$

for all $v \in V(\Omega)$.

Proof. For clarity the proof is divided into four steps.

Step 1: Let the space E^θ be defined by

$$E^\theta(\Omega) = \{v = (v_i) \in L^2(\Omega); \; \tilde{e}_{ij}(v) \in L^2(\Omega)\}. \tag{4.4.20}$$

Then

$$E^\theta(\Omega) = H^1(\Omega). \tag{4.4.21}$$

Let $v = (v_i)$ be an element in $E^\theta(\Omega)$. Then

$$e_{\alpha\beta}(v) = \tilde{e}_{\alpha\beta}(v) + \frac{v_3}{e}(\partial_{\alpha\beta}\theta + x_3\partial_{\alpha\beta}e) \in L^2(\Omega), \;\; e_{i3}(v) = \tilde{e}_{i3}(v). \tag{4.4.22}$$

The identity

$$\partial_{jk}v_i = \partial_j e_{ik}(v) + \partial_k e_{ij}(v) - \partial_i e_{jk}(v)$$

shows that $\partial_{jk}v_i \in H^{-1}(\Omega)$. Also $v \in E^\theta(\Omega) \Rightarrow \partial_j v_i \in H^{-1}(\Omega)$. Hence by lemma of J. L. Lions, we have $\partial_j v_i \in L^2(\Omega)$ and hence $E^\theta(\Omega) \subset H^1(\Omega)$. The reverse inclusion is obvious and hence the equality (4.4.21) follows.

Step 2: The mapping $\|\cdot\|$ defined by

$$\|v\| = \left\{ \|v\|_{0,\Omega} + \sum_{i,j} \|\tilde{e}_{ij}(v)\|^2_{0,\Omega} \right\}^{1/2} \tag{4.4.23}$$

is a norm over the space $H^1(\Omega)$, and there exists a constant C_6 such that

$$\|v\|_{1,\Omega} \leq C_6\|v\| \text{ for all } v \in V(\Omega). \tag{4.4.24}$$

Clearly there exists a constant C_7 such that

$$\|v\| \leq C_7\|v\|_{1,\Omega} \text{ for all } v \in H^1(\Omega). \tag{4.4.25}$$

Hence the identity mapping from the space $H^1(\Omega)$ equipped with the norm $\|\cdot\|_{1,\Omega}$ into the space $E^\theta(\Omega)$ equipped with the norm $\|\cdot\|$ is continuous, and it is also surjective since $E^\theta(\Omega) = H^1(\Omega)$ by the step 1. Since the space $E^\theta(\Omega)$ is a Hilbert space when it is equipped with norm $\|\cdot\|$, the open mapping theorem implies the existence of a constant C_6 satisfying (4.4.24).

Step 3: The semi-norm $|\cdot|^\theta$ is defined by

$$|v|^\theta = \left\{ \sum_{i,j} \|\tilde{e}_{ij}(v)\|^2_{0,\Omega} \right\}^{\frac{1}{2}} \tag{4.4.26}$$

in a norm over the space $V(\Omega)$.

The only property that remains to be checked is that

$$v \in V(\Omega) \text{ and } |v|^\theta = 0 \Rightarrow v = 0.$$

Let $v \in V(\Omega)$ be such that $\tilde{e}_{ij}(v) = 0$. Since $e_{i3}(v) = \tilde{e}_{i3}(v) = 0$, a standard argument (cf. Busse et al. (1997)) implies that there exists functions $\eta_\alpha \in H^1(\omega), \eta_3 \in H^2(\omega), \eta_i = \partial_\nu \eta_3 = 0$ on γ_0 such that $v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3, v_3 = \eta_3$. The relation $\tilde{e}_{\alpha\beta}(v) = 0$ then implies that

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \frac{\eta_3}{e} \partial_{\alpha\beta}\theta = x_3 \left(\partial_{\alpha\beta}\eta_3 + \frac{\eta_3}{e} \partial_{\alpha\beta} e \right)$$

and whence $\partial_{\alpha\beta}\eta_3 + \dfrac{\eta_3}{e}\partial_{\alpha\beta} e = 0$ in ω such that left-hand side of the above equality is only a function of (x_1, x_2).

In particular, $\eta_3 \in H^2(\omega)$ satisfies

$$\Delta \eta_3 + \frac{\eta_3}{e} \Delta e = 0 \ \text{ in } \omega,$$

$$\eta_3 = \partial_\nu \eta_3 = 0 \ \text{ on } \gamma_0. \tag{4.4.27}$$

Let $\omega^{'}$ be a domain which contains γ_0 in its interior. Then the function $\eta_3^{'}$ defined by

$$\eta_3^{'} = \begin{cases} \eta_3 & \text{in } \omega, \\ 0 & \text{in } \omega^{'} - \omega \end{cases} \tag{4.4.28}$$

satisfies $\eta_3^{'} \in H^2(\omega^{'})$,

$$\Delta \eta_3^{'} + \frac{\eta_3^{'}}{e} \Delta e = 0 \ \text{ in } \omega^{'},$$

$$\eta_3^{'} = 0 \text{ in } \omega^{'} - \omega, \tag{4.4.29}$$

and whence $\|\Delta \eta_3^{'}\|_{0,\omega^{'}} \leq C \|\eta_3^{'}\|_{0,\omega^{'}}$ and $\eta_3^{'} = 0$ in $\omega^{'} - \omega$. Hence by Hörmander's theorem, we have $\eta_3^{'} = 0$ in $\omega^{'}$ and hence $\eta_3 = 0$ in ω.

The functions η_α then satisfies $\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha = 0$ in ω, $\eta_\alpha = 0$ in ω, $\eta_\alpha = 0$ on γ_0 and hence $\eta_\alpha = 0$ on ω.

Step 4: There exists a constant C_8 such that

$$\|v\|_{1,\Omega} \leq C_8 |v|^\theta \text{ for all } \ v \in V(\Omega).$$

Suppose the property is false. Then there exists functions $v^k \in V(\Omega), \ k = 1, 2, 3, \cdots$

such that

$$\|v^k\|_{1,\Omega} = 1 \text{ for all } k \geq 1, \quad |v^k|^\theta \to 0 \text{ as } k \to \infty.$$

Since the sequence is bounded in $H^1(\Omega)$, there exists a subsequence $(v)_{l=0}^{\infty}$ that converges strongly in the space $L^2(\Omega)$ by Rellich-Kondrasov theorem. Since $|v^l|^\theta \to 0$ as $l \to \infty$, this subsequence is Cauchy sequence with respect to the norm $\|\cdot\|$. Since this norm is equivalent to the norm $\|\cdot\|_{1,\Omega}$ by step 2, and since the space $H^1(\Omega)$ is complete, the subsequence $(v^l)_{l=1}^{\infty}$ converges in the space $H^1(\Omega)$. On one hand,

$$\|v\|_{1,\Omega} = \lim_{l\to\infty} \|v^l\|_{1,\Omega} = 1. \tag{4.4.30}$$

On the other hand ,

$$|v|^\theta = \lim_{l\to\infty} |v^l|^\theta = 0 \tag{4.4.31}$$

and hence $v = 0$ by step 3, which is impossible by (4.4.30). □

4.5 The Limit Problem

Theorem 4.5.1. *(a) There exists $u \in H^1(\Omega)$, $\varphi \in L^2(\Omega)$ such that*

$$u(\varepsilon) \to u \textit{ in } H^1(\Omega), \quad \varphi(\varepsilon) \to \varphi \textit{ in } L^2(\Omega), \tag{4.5.1}$$

$$(\varepsilon\partial_1\varphi(\varepsilon), \varepsilon\partial_2\varphi(\varepsilon), \partial_3\varphi(\varepsilon)) \to (0, 0, \partial_3\varphi) \textit{ in } L^2(\Omega). \tag{4.5.2}$$

(b) Define the spaces

$$V_H(\omega) = \{(\eta_\alpha) \in (H^1(\omega))^2; \eta_\alpha = 0 \textit{ on } \gamma_0\}, \tag{4.5.3}$$

$$V_3(\omega) = \{\eta_3 \in H^2(\omega); \eta_3 = \partial_\nu\eta_3 = 0 \textit{ on } \gamma_0\}, \tag{4.5.4}$$

$$V_{KL} = \{v \in H^1(\Omega) | v = \eta_\alpha - x_3\partial_\alpha\eta_3, (\eta_i) \in V_H(\omega) \times V_3(\omega)\}, \tag{4.5.5}$$

$$\Psi_l = \{\psi \in L^2(\Omega), \partial_3\psi \in L^2(\Omega)\}, \tag{4.5.6}$$

$$\Psi_{l0} = \{\psi \in L^2(\Omega); \partial_3\psi \in L^2(\Omega), \psi|\Gamma^\pm = 0\} \tag{4.5.7}$$

Then there exists $(\zeta_\alpha, \zeta_3) \in V_H(\omega) \times V_3(\omega)$ *such that*

$$u_\alpha = \zeta_\alpha - x_3 \partial_\alpha \zeta_3 \quad \textit{and} \quad u_3 = \zeta_3, \tag{4.5.8}$$

$$\varphi = \sum_{i=0}^{2} \varphi^m(x_1, x_2) x_3^m, \tag{4.5.9}$$

where

$$\begin{aligned} \varphi^0 = \frac{\varphi_0^+ + \varphi_0^-}{2} + \frac{p^{3\alpha\beta}}{2p^{33}}(\partial_{\alpha\beta}\zeta_3 + \frac{\partial_{\alpha\beta} e}{e}\zeta_3),\ \varphi_0^\pm = \varphi_{|\Gamma^\pm}, \\ \varphi^1 = \frac{\varphi_0^+ - \varphi_0^-}{2},\ \varphi^2 = \frac{p^{3\alpha\beta}}{2p^{33}}(\partial_{\alpha\beta}\zeta_3 + \frac{\partial_{\alpha\beta} e}{e}\zeta_3). \end{aligned} \tag{4.5.10}$$

and $(\zeta) = (\zeta_\alpha, \zeta_3) \in V_H(\omega) \times V_3(\omega)$ *satisfies*

$$\begin{aligned} &-\int_\omega m_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\eta_3 e d\omega - \int_\omega \left[n^\theta_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\theta + m_{\alpha\beta}(\zeta)\partial_{\alpha\beta} e \right] \eta_3 e d\omega \\ &+ \int_\omega n^\theta_{\alpha\beta}\partial_\beta \eta_\alpha e d\omega + \frac{2}{3}\int_\omega \frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3 \partial_{\alpha\beta}\eta_3 e d\omega \\ &\quad = \int_\omega p^i \eta_i e d\omega - \int_\omega q^\alpha \partial_\alpha \eta_3 e d\omega - \int_\omega \frac{\varphi^+ - \varphi^-}{2} p^{3\alpha\beta}\hat{e}_{\alpha\beta}(\eta) e d\omega \end{aligned} \tag{4.5.11}$$

where

$$m_{\alpha\beta}(\zeta) = -\frac{4\lambda\mu}{3(\lambda + 4\mu)}\left(\triangle\zeta_3 + \zeta_3 \frac{\triangle e}{e}\right)\delta_{\alpha\beta} + \frac{4\mu}{3}\left(\partial_{\alpha\beta}\zeta_3 + \zeta_3 \frac{\partial_{\alpha\beta} e}{e}\right), \tag{4.5.12}$$

$$n^\theta_{\alpha\beta}(\zeta) = \frac{4\lambda\mu}{\lambda + 2\mu}\hat{e}_{\sigma\sigma}(\zeta)\delta_{\alpha\beta} + 4\mu \hat{e}_{\alpha\beta}(\zeta), \tag{4.5.13}$$

$$\hat{e}_{\alpha\beta}(\zeta) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha) - \zeta_3\frac{\partial_{\alpha\beta}\theta}{e} = \frac{1}{2}\int_{-1}^{1}\tilde{e}_{\alpha\beta}(\zeta)dx_3, \tag{4.5.14}$$

$$p^{33} = \frac{e}{\mu}P^{3\alpha 3}P^{3\alpha 3} + \frac{e^4}{\lambda + 2\mu}P^{333}P^{333} + \in^{33}, \tag{4.5.15}$$

$$p^{3\alpha\beta} = P^{3\alpha\beta} - \frac{\lambda e^2}{\lambda + 2\mu}P^{333}\delta^{\alpha\beta}, \tag{4.5.16}$$

$$p^i = \int_{-1}^{1} f(\cdot, x_3)dx_3,\ \ q^\alpha = \int_{-1}^{1} x_3 f^\alpha dx_3. \tag{4.5.17}$$

Proof. For the sake of clarity, the proof is divided into several steps.

Step 1: Define the vector $\tilde{\varphi}_i(\varepsilon)$ by

$$\tilde{\varphi}(\varepsilon) = (\tilde{\varphi}_i(\varepsilon)) = (\varepsilon\partial_1\overline{\varphi}(\varepsilon), \varepsilon\partial_2\overline{\varphi}(\varepsilon), \partial_3\overline{\varphi}(\varepsilon)). \tag{4.5.18}$$

Then there exists constant $C_9 > 0$ and $\varepsilon_0 > 0$ such that

$$||u(\varepsilon)||_{1,\Omega} \leq C_9, \quad |\tilde{\varphi}_i(\varepsilon)| \leq C_9 \tag{4.5.19}$$

for all $0 < \varepsilon \leq \varepsilon_0$.

Letting $(v, \psi) = (u(\varepsilon), \varphi(\varepsilon))$ in (4.3.12), we get

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon, u(\varepsilon))e_{i||j}(\varepsilon, u(\varepsilon))\sqrt{g(\varepsilon)}dx + \int_\Omega \in^{ij}(\varepsilon)\tilde{\varphi}_i(\varepsilon)\tilde{\varphi}_j(\varepsilon)\sqrt{g(\varepsilon)}dx \\
&= \int_\Omega f \cdot u(\varepsilon)\sqrt{g(\varepsilon)}dx - \int_\Omega \in^{ij}(\varepsilon)\tilde{\varphi}_{i0}(\varepsilon)\tilde{\varphi}_j(\varepsilon)\sqrt{g(\varepsilon)}dx \\
&- \int_\Omega P^{mij}(\varepsilon)\tilde{\varphi}_{m0}(\varepsilon)e_{i||j}(\varepsilon, u(\varepsilon))\sqrt{g(\varepsilon)}dx.
\end{aligned} \tag{4.5.20}$$

Using the coerciveness properties (4.2.28) and (4.2.29), the inequality $(a-b)^2 \geq a^2/2 - b^2$ and the inequality (4.4.19), for $\varepsilon \leq min\{\varepsilon_0, 1\}$, we have

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon, u(\varepsilon))e_{i||j}(\varepsilon, u(\varepsilon))\sqrt{g(\varepsilon)}dx + \int_\Omega \in^{ij}(\varepsilon)\tilde{\varphi}_i(\varepsilon)\tilde{\varphi}_j(\varepsilon)\sqrt{g(\varepsilon)}dx \\
&\geq C_{10}\sum_{i,j}\left\|e_{i||j}(\varepsilon, u(\varepsilon))\right\|_{0,\Omega}^2 + C_{10}\sum_i \|\tilde{\varphi}_i(\varepsilon)\|_{0,\Omega}^2 \\
&= C_{10}\sum_{\alpha,\beta}\left\|\tilde{e}_{\alpha\beta}(u(\varepsilon)) + \varepsilon^2 e^\sharp_{\alpha\beta}(\varepsilon, u(\varepsilon))\right\|_{0,\Omega}^2 + 2C_{10}\sum_\alpha\left\|\frac{1}{\varepsilon}\tilde{e}_{\alpha 3}(u(\varepsilon)) + \varepsilon e^\sharp_{\alpha 3}(\varepsilon, u(\varepsilon))\right\|_{0,\Omega}^2 \\
&\quad + C_{10}\left\|\frac{1}{\varepsilon^2}\tilde{e}_{33}(u(\varepsilon))\right\|_{0,\Omega}^2 + C_{10}\sum_i \|\tilde{\varphi}_i(\varepsilon)\|_{0,\Omega}^2 \\
&\geq C_{10}\left\{\frac{1}{2}\sum_{i,j}\|\tilde{e}_{ij}(u(\varepsilon))\|_{0,\Omega}^2 - 3\varepsilon^3 C_{10}^2\|u(\varepsilon)\|_{1,\Omega}^2\right\} + C_{10}\sum_i \|\tilde{\varphi}_i(\varepsilon)\|_{0,\Omega}^2 \\
&\geq C_{10}\left\{\frac{1}{2}(C_5)^{-2} - 3\varepsilon^2 C_{10}^2\right\}\|u(\varepsilon)\|_{1,\Omega}^2 + C_{10}\sum_i \|\tilde{\varphi}_i(\varepsilon)\|_{0,\Omega}^2 \\
&\geq C_{11}\left(\|u(\varepsilon)\|_{0\Omega}^2 + \|\tilde{\varphi}(\varepsilon)\|_{0,\Omega}^2\right).
\end{aligned} \tag{4.5.21}$$

Also

$$
\begin{aligned}
&\int_\Omega f \cdot u(\varepsilon)\sqrt{g(\varepsilon)}dx - \int_\Omega \in^{ij} (\varepsilon)\tilde{\varphi}_{i0}(\varepsilon)\tilde{\varphi}_j(\varepsilon)\sqrt{g(\varepsilon)}dx \\
&\quad - \int_\Omega P^{mij}(\varepsilon)\tilde{\varphi}_{m0}(\varepsilon)e_{i\|j}(\varepsilon, u(\varepsilon))\sqrt{g(\varepsilon)}dx \\
&\qquad \leq C\left(\|u(\varepsilon)\|_{1,\Omega} + \|\tilde{\varphi}(\varepsilon)\|_{0,\Omega}\right).
\end{aligned} \tag{4.5.22}
$$

Relation (4.5.19) follows from (4.5.21) and (4.5.22).

Step 2: From step 1 it follows that there exists a subsequence $(\tilde{\varphi}(\varepsilon))$ and $\tilde{\varphi} \in L^2(\Omega)$ such that

$$
(\varepsilon\partial_1\overline{\varphi}(\varepsilon), \varepsilon\partial_2\overline{\varphi}(\varepsilon), \partial_3\overline{\varphi}(\varepsilon)) \rightharpoonup (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3) \text{ in } (L^2(\Omega))^3. \tag{4.5.23}
$$

Since Γ_{hD} contains Γ^-, we have,

$$
\overline{\varphi}(\varepsilon)(x_1, x_2, x_3) = \int_{-1}^{x_3} \partial_3\overline{\varphi}(\varepsilon)(x_1, x_2, s)ds \tag{4.5.24}
$$

and it follows that $\|\overline{\varphi}(\varepsilon)\|_{0,\Omega} \leq \sqrt{2}\,\|\partial_3\overline{\varphi}(\varepsilon)\|_{0,\Omega}$. This implies that $\varphi(\varepsilon)$ is bounded in $L^2(\Omega)$. Therefore there exists a φ in $L^2(\Omega)$ and a subsequence, still indexed by ε, such that $\varphi(\varepsilon)$ converges weakly to φ. Hence it follows from (4.5.23) that

$$
(\varepsilon\partial_1\varphi(\varepsilon), \varepsilon\partial_2\varphi(\varepsilon), \partial_3\varphi(\varepsilon)) \rightharpoonup (0, 0, \partial_3\varphi). \tag{4.5.25}
$$

Step 3: Define the tensor $\tilde{K} = (\tilde{K}_{ij})$ by

$$
\tilde{K}_{\alpha\beta}(\varepsilon) = \tilde{e}_{\alpha\beta}(u(\varepsilon)),\ \ \tilde{K}_{\alpha 3}(\varepsilon) = \frac{1}{\varepsilon}\tilde{e}_{\alpha 3}(u(\varepsilon)),\ \ \tilde{K}_{33}(\varepsilon) = \frac{1}{\varepsilon^2}\tilde{e}_{33}(u(\varepsilon)). \tag{4.5.26}
$$

Then there exists a constant C_{12} such that

$$
\left\|\tilde{K}(\varepsilon)\right\|_{0,\Omega} \leq C_{12}, \quad for\ \ all\ \ 0 < \varepsilon < \varepsilon_0. \tag{4.5.27}
$$

Using the definition (4.5.26) and the relations (4.4.1)- (4.4.3)

$$\begin{aligned}\left\|\tilde{K}(\varepsilon)\right\|_{0,\Omega} &= \sum_{\alpha,\beta}\left\|e_{\alpha||\beta}(\varepsilon,u(\varepsilon)) - \varepsilon^2 e^{\sharp}(\varepsilon,u(\varepsilon))\right\|_{0,\Omega}^2 \\ &= 2\sum_{\alpha}\left\|e_{\alpha||3}(\varepsilon,u(\varepsilon)) - \varepsilon e^{\sharp}_{\alpha||3}(\varepsilon,u(\varepsilon))\right\|_{0,\Omega}^2 + \left\|e_{3||3}(\varepsilon,u(\varepsilon))\right\|_{0,\Omega}^2 \\ &\leq 2\sum_{i,j}\left\|e_{i||j}(\varepsilon,u(\varepsilon))\right\|_{0,\Omega}^2 + 2\varepsilon^4\Sigma_{\alpha,\beta}\left\|e^{\sharp}(\varepsilon,u(\varepsilon))\right\|_{0,\Omega}^2 + 4\varepsilon^2\sum_{\alpha}\left\|e^{\sharp}_{\alpha,3}(\varepsilon,u(\varepsilon))\right\|_{0,\Omega}^2.\end{aligned} \tag{4.5.28}$$

Hence the relation (4.5.27) follows by using the inequalities (4.4.7) and (4.5.19).

Step 4: From step 1 it follows that there exists a subsequence $(u(\varepsilon))$ and a function $u \in V$ such that

$$u(\varepsilon) \rightharpoonup u \quad \text{in} \quad H^1(\Omega) \quad \text{as} \quad \varepsilon \to 0.$$

Then there exist functions $(\zeta_\alpha) \in H^1(\omega)$ and $\zeta_3 \in H^2(\omega)$ satisfying $\zeta_i = \partial_\nu\zeta_3 = 0.$ on γ_0 such that

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3 \text{ and } u_3 = \zeta_3. \tag{4.5.29}$$

From the definition (4.5.26) and the boundedness of $(\tilde{K}_{ij}(\varepsilon))$, we deduce that

$$\|e_{\alpha 3}(u(\varepsilon))\|_{0,\Omega} \leq \varepsilon C_{13} \text{ and } \|e_{33}(u(\varepsilon))\|_{0,\Omega} \leq \varepsilon^2 C_{13}$$

where $e_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. Since norm is a weakly lower semicontinuous function

$$\|e_{i3}(u)\|_{0,\Omega} \leq \liminf_{\varepsilon\to 0}\|e_{i3}(u(\varepsilon)\|_{0,\Omega} = 0 \tag{4.5.30}$$

hence $e_{i3}(u) = 0$. Then it is a standard argument that the components u_i of the limit u are of the form (4.5.29).

Step 5: From step 3 there exists a subsequence and an element $\tilde{K} = (\tilde{K}_{ij}) \in L^2(\Omega)$ such that

$$\tilde{K}(\varepsilon) \rightharpoonup \tilde{K} \text{ in } L^2(\Omega) \text{ as } \varepsilon \to 0. \tag{4.5.31}$$

Then

$$\tilde{K}_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u),\ \ \tilde{K}_{\alpha 3} = -\frac{e}{\mu}P^{3\alpha 3}\partial_3\varphi,\ \ \tilde{K}_{33} = -\frac{e^2}{\lambda+2\mu}(e^2P^{333}\partial_3\varphi+\lambda\tilde{K}_{\beta\beta}) \tag{4.5.32}$$

Since $u(\varepsilon) \rightharpoonup u$ in $H^1(\Omega)$, the definition (4.4.4) of the functions $\tilde{e}_{\alpha\beta}(v)$ shows that the function $\tilde{K}_{\alpha\beta}(\varepsilon) = \tilde{e}_{\alpha\beta}(u(\varepsilon))$ converges weakly in $L^2(\Omega)$ to the function $\tilde{e}_{\alpha\beta}(u)$.

We next recall the following result. Let $w \in L^2(\Omega)$ be given. Then

$$\int_\Omega w\partial_3 v dx = 0 \text{ for all } v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_0,\ \text{ then } w = 0. \tag{4.5.33}$$

The equation (4.3.12) can be written as

$$\begin{aligned}
&\int_\Omega \Big(\Big\{\Big[A^{\alpha\beta\sigma\tau}(0)+\varepsilon^2A_\sharp^{\alpha\beta\sigma\tau}(\varepsilon)\Big]\Big[\tilde{K}_{\sigma\tau}(\varepsilon)+\varepsilon^2e^\sharp_{\sigma\tau}(\varepsilon;u(\varepsilon))\Big]+\Big[A^{\alpha\beta 33}(0)+\varepsilon^2A_\sharp^{\alpha\beta 33}(\varepsilon)\Big]\tilde{k}_{33}(\varepsilon)\Big\}\\
&\Big\{\frac{1}{2}\partial_\alpha v_\beta+\frac{1}{2}\partial_\beta v_\alpha-\frac{v_3}{e}(\partial_{\alpha\beta}\theta+x_3\partial_{\alpha\beta}e)+\varepsilon^2e^\sharp_{\alpha\beta}(\varepsilon;v)\Big\}\\
&+\{4[A^{\alpha 3\sigma 3}(0)+\varepsilon^2A_\sharp^{\alpha 3\sigma 3}(\varepsilon)][\tilde{K}(\varepsilon)_{\sigma 3}+\varepsilon e^\sharp_{\sigma 3}(\varepsilon;u(\varepsilon))]\}\\
&\Big\{\frac{1}{2\varepsilon}\partial_\alpha v_3+\frac{1}{2\varepsilon}\partial_3 v_\alpha+\varepsilon e^\sharp_{\alpha 3}(\varepsilon;v)\Big\}+\Big\{\big[A^{33\sigma\tau}(0)+\varepsilon^2A_\sharp^{33\sigma\tau}(\varepsilon)\big]\Big[\tilde{K}_{\sigma\tau}(\varepsilon)+\varepsilon^2e^\sharp_{\sigma\tau}(\varepsilon;u(\varepsilon))\Big]\\
&+\big[A^{3333}(0)+\varepsilon^2A_\sharp^{3333}(\varepsilon)\big]\tilde{K}_{33}(\varepsilon)\Big\}\Big\{\frac{1}{\varepsilon^2}\partial_3 v_3\Big\}\Big)\sqrt{e^2+\varepsilon^2g^\sharp(\varepsilon)}\,dx\\
&+\int_\Omega \in^{33}(\varepsilon)\partial_3\overline{\varphi}(\varepsilon)\partial_3\psi\sqrt{g(\varepsilon)}dx+\int_\Omega P^{3kl}\left[\partial_3\overline{\varphi}(\varepsilon)e_{k||l}(\varepsilon,v)-\partial_3\psi e_{k||l}(\varepsilon,u(\varepsilon))\right]\sqrt{g(\varepsilon)}dx\\
&+\varepsilon\int_\Omega \in^{3\alpha}(\varepsilon)\left[\partial_\alpha\overline{\varphi}(\varepsilon)\partial_3\psi+\partial_3\overline{\varphi}(\varepsilon)\partial_\alpha\psi)\right]\sqrt{g(\varepsilon)}dx+\varepsilon^2\int_\Omega \in^{\alpha\beta}(\varepsilon)\partial_\alpha\overline{\varphi}(\varepsilon)\partial_\beta\psi\sqrt{g(\varepsilon)}dx\\
&+\varepsilon\int_\Omega\left[P^{\alpha kl}(\varepsilon)\partial_\alpha\overline{\varphi}(\varepsilon)e_{k||l}(\varepsilon,u(\varepsilon))-\partial_\alpha\psi e_{k||l}(\varepsilon,v)\right]\sqrt{g(\varepsilon)}dx\\
&=\int_\Omega f^iv_i\sqrt{e^2+\varepsilon^2g^\sharp(\varepsilon)}\,dx-\int_\Omega \in^{33}(\varepsilon)\partial_3\varphi_0\partial_3\psi\sqrt{g(\varepsilon)}\,dx\\
&-\varepsilon\int_\Omega \in^{3\alpha}(\varepsilon)\left[(\partial_\alpha\varphi_0\partial_3\psi+\partial_3\varphi_0\partial_\alpha\psi)\right]\sqrt{g(\varepsilon)}\,dx+\varepsilon^2\int_\Omega \in^{\alpha\beta}(\varepsilon)\partial_\alpha\varphi_0\partial_\beta\psi\sqrt{g(\varepsilon)}\,dx\\
&-\int_\Omega P^{3kl}(\varepsilon)\partial_3\varphi_0e_{k||l}(\varepsilon,v)\sqrt{g(\varepsilon)}\,dx-\varepsilon\int_\Omega P^{\alpha ij}(\varepsilon)\partial_3\varphi_0e_{i||j}(\varepsilon)(v)\sqrt{g(\varepsilon)}\,dx\ \forall\, v\in V(\Omega).
\end{aligned} \tag{4.5.34}$$

Multiplying the above equation by ε^2, taking $v_\alpha = 0$ and letting $\varepsilon \to 0$, we get

$$\int_\Omega \left[\frac{\lambda}{e^2} \tilde{K}_{\sigma\sigma} + \frac{(\lambda + 2\mu)}{e^4} \tilde{K}_{33} + P^{333} \partial_3 \varphi \right] \partial_3 v_3 e dx = 0 \tag{4.5.35}$$

which implies $e^2 \lambda \tilde{K}_{\sigma\sigma} + (\lambda + 2\mu) \tilde{K}_{33} + e^4 P^{333} \partial_3 \varphi = 0$ and hence the third relation in (4.5.32) follows.

Again, multiplying equation (4.5.34) by ε, taking $v_3 = 0$ and letting $\varepsilon \to 0$, we get

$$\int_\Omega \left[\frac{\mu}{e} \tilde{K}_{\alpha 3} + P^{3\alpha 3} \partial_3 \varphi \right] \partial_3 v_\alpha dx = 0 \tag{4.5.36}$$

which implies $(\mu \tilde{K}_{\alpha 3} + e P^{3\alpha 3} \partial_3 \varphi) = 0$ and hence the second relation in (4.5.32) follows.

Step 6: The function φ is of the form (4.5.9).

Using the scalings (4.3.8) we have

$$D_i(\varepsilon)(u(\varepsilon), \varphi(\varepsilon)) = P^{ikl}(\varepsilon)\; e_{k||l}(\varepsilon, u(\varepsilon)) + \in^{ij} (\varepsilon) E_j(\varepsilon)(\varphi(\varepsilon)). \tag{4.5.37}$$

Passing to the limit, we get

$$\lim_{\varepsilon \to 0} D_i(\varepsilon)(u(\varepsilon), \varphi(\varepsilon)) = D_i = P^{ikl} \tilde{K}_{kl} - \in^{i3} \partial_3 \varphi. \tag{4.5.38}$$

In particular

$$\begin{aligned}
\lim_{\varepsilon \to 0} D_3(\varepsilon)(u(\varepsilon), \varphi(\varepsilon)) = D_3 &= P^{3kl} \tilde{K}_{kl} - \in^{33} \partial_3 \varphi \\
&= \left[P^{3\alpha 3} - \frac{\lambda e^2}{\lambda + 2\mu} \delta^{\alpha\beta} P^{333} \right] \tilde{K}_{\alpha\beta} \\
&= \left[\frac{e P^{3\alpha 3} P^{3\alpha 3}}{\mu} + \frac{e^4 P^{333} P^{333}}{\lambda + 2\mu} + \in^{33} \right] \partial_3 \varphi \\
&= p^{3\alpha\beta} \tilde{K}_{\alpha\beta} - p^{33} \partial_3 \varphi.
\end{aligned} \tag{4.5.39}$$

Taking $v = 0$ and letting $\varepsilon \to 0$ in equation (4.3.12), we get

$$\int_\Omega D_3(u, \varphi)\partial_3\varphi e dx = 0. \tag{4.5.40}$$

$$i.e., \quad \int_\Omega \left(p^{3\alpha\beta}\tilde{K}_{\alpha\beta} - p^{33}\partial_3\varphi\right)\partial_3\psi e dx = 0. \tag{4.5.41}$$

Since $\mathcal{D}(\Omega)$ is dense in Ψ_{l0} for the norm $||.||_{\Psi_l}$, equation (4.5.41) is equivalent to

$$\partial_3(p^{3\alpha\beta}\tilde{K}_{\alpha\beta} - p^{33}\partial_3\varphi) = 0 \text{ in } \mathcal{D}'(\Omega) \tag{4.5.42}$$

which implies that $(p^{3\alpha\beta}\tilde{K}_{\alpha\beta} - p^{33}\partial_3\varphi) = d^1$, with $d^1 \in \mathcal{D}'(\omega)$. In fact, due to the regularity of u and φ, d^1 is in $L^2(\omega)$. Then

$$\partial_3\varphi = \frac{p^{3\alpha\beta}}{p^{33}}\left[\hat{e}_{\alpha\beta}(\zeta) - x_3\left(\partial_{\alpha\beta}\zeta_3 + \frac{\partial_{\alpha\beta}e}{e}\zeta_3\right)\right] - \frac{1}{p^{33}}d^1 \tag{4.5.43}$$

which gives

$$\varphi = \frac{p^{3\alpha\beta}}{p^{33}}\left[x_3\hat{e}_{\alpha\beta}(\zeta) - x_3^2\left(\partial_{\alpha\beta}\zeta_3 + \frac{\partial_{\alpha\beta}e}{e}\zeta_3\right)\right] - \frac{x_3}{p^{33}}d^1 + d^0. \tag{4.5.44}$$

Since φ satisfies the boundary conditions $\varphi|_{\Gamma^+} = \varphi_0^+, \ \varphi|_{\Gamma^-} = \varphi_0^-$, we have

$$d^0 = \frac{\varphi_0^+ + \varphi_0^-}{2} + \frac{p^{3\alpha\beta}}{2p^{33}}\left(\partial_{\alpha\beta}\zeta_3 + \frac{\partial_{\alpha\beta}e}{e}\zeta_3\right), \tag{4.5.45}$$

$$d^1 = p^{3\alpha\beta}\tilde{e}_{\alpha\beta}(\zeta) - p^{33}\frac{\varphi_0^+ - \varphi_0^-}{2}. \tag{4.5.46}$$

Thus the conclusion follows.

Step 7: The function (ζ_i) satisfies (4.5.11).

Taking $\psi = 0$ and $v \in V_{KL}$ and letting $\varepsilon \to 0$ in equation (4.5.34) we have

$$\int_\Omega A^{\alpha\beta kl} \tilde{K}_{kl} \tilde{K}_{\alpha\beta}(v) e dx + \int_\Omega P^{3\alpha\beta} \partial_3 \varphi \tilde{K}_{\alpha\beta}(v) e dx$$
$$= \int_\Omega f \cdot v e dx - \int_\Omega P^{3\alpha\beta} \partial_3 \varphi_0 \tilde{K}_{\alpha\beta}(v) e dx. \quad (4.5.47)$$

Replacing u and $\tilde{K}_{ij}$ by the expressions obtained in (4.5.29) and (4.5.32), and taking v of the form

$$v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3 \text{ and } v_3 = \eta_3 \quad (4.5.48)$$

with $(\eta_i) \in V(\omega)$, it is verified that equation (4.5.47) coincide with equation (4.5.11).

Step 8: To prove the uniqueness of the solution, let $B(\zeta, \eta)$ denote the bilinear form associated with the left hand side of (4.5.11) and $L(\eta)$ denote the linear form associated with the right hand side of (4.5.11) . It has been proved in Sabu (2001) that for $\eta \in V(\omega)$ there exists a constant C such that

$$- \int_\omega m_{\alpha\beta}(\eta) \partial_{\alpha\beta} \eta_3 e d\omega - \int_\omega \left[n^\theta_{\alpha\beta}(\eta) \partial_{\alpha\beta} \theta + m_{\alpha\beta}(\eta) \partial_{\alpha\beta} e \right] \eta_3 e d\omega$$
$$+ \int_\omega n^\theta_{\alpha\beta} \partial_\beta \eta_\alpha e d\omega \geq C\{||\eta_\alpha||^2_{1,\omega} + ||\eta_3||^2_{2,\omega}\} \quad (4.5.49)$$

Hence

$$\begin{aligned} B(\eta, \eta) \quad &\geq \quad C\{||\eta_\alpha||^2_{1,\omega} + ||\eta_3||^2_{2,\omega}\} + \frac{2}{3} \left|\left| \frac{p^{3\alpha\beta}}{\sqrt{p^{33}}} \partial_{\alpha\beta} \eta_3 \right|\right|^2_{0,\omega} \\ &\geq \quad C\{||\eta_\alpha||^2_{1,\omega} + ||\eta_3||^2_{2,\omega}\} \end{aligned} \quad (4.5.50)$$

Hence the bilinear form $B(\cdot, \cdot)$ is elliptic. Since the linear form $L(\eta)$ is continuous, the uniqueness of solution follows.

Step 9: The strong convergences of $u(\varepsilon) \rightharpoonup u$ in $H^1(\Omega)$ and $\varphi(\varepsilon) \rightharpoonup \varphi$ in $L^2(\Omega)$ can be proved as in Busse et al. (1997). □

Theorem 4.5.2. *A smooth enough solution* (ζ_i) *of the variational problem* (4.5.11) *is* *also a solution of the following two dimensional boundary value problem:*

$$-\partial_{\alpha\beta}(em_{\alpha\beta}) - [n^{\theta}(\zeta)\partial_{\alpha\beta}\theta + m_{\alpha\beta}(\zeta)\partial_{\alpha\beta}e]e + \frac{2}{3}\partial_{\alpha\beta}\left(e\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\right)$$

$$= p^3 e + \partial_\alpha(eq^\alpha) + \frac{\varphi^+ - \varphi^-}{2}p^{3\alpha\beta}\partial_{\alpha\beta}\theta \text{ in } \omega, \quad (4.5.51)$$

$$-\partial_\beta(en^{\theta}_{\alpha\beta}) = p^\alpha e + \partial_\beta(e\frac{\varphi^+ - \varphi^-}{2}p^{3\alpha\beta}) \text{ in } \omega, \quad (4.5.52)$$

$$\zeta_i = \partial_\nu\zeta_3 = 0 \text{ on } \gamma_0, \quad (4.5.53)$$

$$\partial_\alpha(em_{\alpha\beta})\nu_\beta + \partial_\tau(em_{\alpha\beta}\nu_\alpha\tau_\beta) + \frac{2}{3}\partial_\alpha\left(e\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\right)\nu_\beta$$

$$+ \frac{2}{3}\partial_\tau\left(e\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\nu_\alpha\tau_\beta\right) = -eq^\alpha\nu_\alpha \text{ on } \gamma_1, \quad (4.5.54)$$

$$\left(m_{\alpha\beta} + \frac{2}{3}\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\right)\nu_\alpha\nu_\beta = 0 \text{ on } \gamma_1, \quad (4.5.55)$$

$$n^{\theta}_{\alpha\beta}\nu_\beta = \left(\frac{\varphi^+ - \varphi^-}{2}p^{3\alpha\beta}\right)\nu_\beta \text{ on } \gamma_1. \quad (4.5.56)$$

Proof. Applying the *Green's formula:*

$$\int_\omega \varphi\partial_\alpha\chi \, d\omega = -\int_\omega (\partial_\alpha\varphi)\chi \, d\omega + \int_\gamma \varphi\chi\nu_\alpha \, d\gamma$$

we get

$$-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 \, ed\omega = -\int_\omega \partial_{\alpha\beta}(em_{\alpha\beta})\eta_3 \, d\omega + \int_\gamma \partial_\alpha(em_{\alpha\beta})\nu_\beta\eta_3 \, d\gamma - \int_\gamma em_{\alpha\beta}\nu_\alpha\partial_\beta\eta_3 \, d\gamma.$$

Since $\partial_\beta\eta_3 = \nu_\beta\partial_\nu\eta_3 + \tau_\beta\partial_\tau\eta_3$, we have

$$\int_\gamma em_{\alpha\beta}\nu_\alpha\partial_\beta\eta_3 \, d\gamma = \int_\gamma em_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 \, d\gamma + \int_\gamma em_{\alpha\beta}\nu_\alpha\tau_\beta\partial_\tau\eta_3 \, d\gamma.$$

Observing that

$$\int_\gamma \varphi\partial_\tau\eta_3 \, d\gamma = -\int_\gamma (\partial_\tau\varphi)\eta_3 \, d\gamma \quad \text{since} \quad \int_\gamma \partial_\tau(\varphi\eta_3) \, d\gamma = 0,$$

we obtain

$$-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3\, ed\omega \;=\; -\int_\omega \partial_{\alpha\beta}(em_{\alpha\beta})\eta_3\, d\omega + \int_\gamma \{\partial_\alpha(em_{\alpha\beta})\nu_\beta + \partial_\tau(em_{\alpha\beta}\nu_\alpha\tau_\beta)\}\eta_3\, d\gamma$$
$$-\int_\gamma em_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3\, d\gamma \qquad (4.5.57)$$

Similarly we have

$$\int_\omega \frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\partial_{\alpha\beta}\eta_3 ed\omega = -\int_\omega \partial_{\alpha\beta}\left(e\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\right)\eta_3\, d\omega$$
$$+\int_\gamma \left\{\partial_\alpha\left(e\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\right)\nu_\beta\right.$$
$$\left.+\,\partial_\tau\left(e\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\nu_\alpha\tau_\beta\right)\right\}\eta_3\, d\gamma$$
$$-\int_\gamma e\frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\nu_\alpha\nu_\beta\partial_\nu\eta_3\, d\gamma \qquad (4.5.58)$$

$$\int_\omega n^\theta_{\alpha\beta}\partial_\beta\eta_\alpha\, ed\omega = -\int_\omega \partial_\beta(en^\theta_{\alpha\beta})\eta_\alpha\, d\omega + \int_\gamma en^\theta_{\alpha\beta}\nu_\beta\eta_\alpha\, d\gamma, \qquad (4.5.59)$$

$$-\int_\omega q^\alpha\partial_\alpha\eta_3\, ed\omega = \int_\omega \partial_\alpha(eq^\alpha)\eta_3\, d\omega - \int_\gamma eq^\alpha\nu_\alpha\eta_3\, d\gamma \qquad (4.5.60)$$

$$\int_\omega \frac{\varphi^+ - \varphi^-}{2}p^{3\alpha\beta}\hat{e}_{\alpha\beta}(\eta)ed\omega = -\int_\omega \partial_\beta\left(e\frac{\varphi^+ - \varphi^-}{2}p^{3\alpha\beta}\right)\eta_\alpha d\omega$$
$$+\int_\gamma \left(e\frac{\varphi^+ - \varphi^-}{2}p^{3\alpha\beta}\right)\nu_\beta\eta_\alpha d\gamma$$
$$-\int_\omega \frac{\varphi^+ - \varphi^-}{2}p^{3\alpha\beta}\partial_{\alpha\beta}\theta\eta_3 d\omega \qquad (4.5.61)$$

Combining the above equations with the boundary conditions $\eta_i = \partial_\nu\eta_3 = 0$ on γ_0 gives the boundary value problem stated in the theorem. □

CONCLUDING REMARKS

- We have rigorously justified two-dimensional linearly elastic shallow shells model using gamma convergence.
- We justified the scalings used in Busse et al. (1997) to derive the two dimensional shallow shell model.
- We consider a thin piezoelectric shallow shell with *variable* thickness and we have shown that under suitable scalings on the data, as the thickness of the shell goes to zero, the solution of the three-dimensional piezoelectric shell converge to the solution of two-dimensional model of piezoelectric shallow shell with variable thickness.

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1. Raja, J., and Sabu, N. (2013). Justification of two dimensional model of shallow shells using gamma convergence. *Indian Journal of Pure and Applied Mathematics*, 44(3), 277-295.

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